7. Coriolis force

21. November 2024

Problem 1.
Evaluate how approximately the trajectory of a projectile shot to the east in the northern hemisphere changes due to the Coriolis effect. Assume that the distance covered by the projectile is small compared to the Earth radius. Use the Cartesian system with the axis x pointing to the east, y to the north and z upward. Consider the approximation with the forces computed from the velocity of the original parabolic trajectory only (without effects of the Coriolis force).

Solution:
In the given coordinates, the angular velocity vector of the Earth rotation is

$$\boldsymbol{\Omega} = (0, \Omega\cos\varphi, \Omega\sin\varphi)\,,$$

where $\Omega = 2\pi/24hod$ is the size of the angular velocity and φ is the latitude. Given the assumption that the trajectory of the projectile is short, we can consider $\varphi = konst$.

If we do not consider the Coriolis force, the velocity of the projectile would be

$$\mathbf{v}_p = (V_0\cos\alpha, 0, V_0\sin\alpha - gt),$$

where V_0 is the size of the initial velocity, α is the angle under which it was shot, g is the acceleration by the gravity and t is the elapsed time after the shot.

The Coriolis force is than given by the formula

$$\mathbf{F}_{cor} = -2m\boldsymbol{\Omega}\times\mathbf{v}_p = -2m\Omega\begin{pmatrix}\cos\varphi\,(V_0\sin\alpha - gt)\\ V_0\sin\varphi\cos\alpha\\ -V_0\cos\varphi\cos\alpha.\end{pmatrix}$$

The total acceleration caused by the Coriolis force and the gravity is therefore

$$\mathbf{a} = \begin{pmatrix}2\Omega gt\cos\varphi\\ 0\\ -g\end{pmatrix} + 2\Omega V_0\begin{pmatrix}-\cos\varphi\sin\alpha\\ -\sin\varphi\cos\alpha\\ \cos\varphi\cos\alpha\end{pmatrix}.$$

Because the acceleration does not depend on the location in the approximation, we can get velocity by integrating the acceleration over time:

$$\mathbf{v}(t) = \mathbf{v}_p(0) + \int_0^t \mathbf{a}(s)\,\mathrm{d}s = \begin{pmatrix}V_0\cos\alpha\\ 0\\ V_0\sin\alpha\end{pmatrix} + \begin{pmatrix}\Omega gt^2\cos\varphi\\ 0\\ -gt\end{pmatrix} + 2\Omega V_0 t\begin{pmatrix}-\cos\varphi\sin\alpha\\ -\sin\varphi\cos\alpha\\ \cos\varphi\cos\alpha\end{pmatrix}.$$

If the initial position of the projectile was zero, subsequent integration gives the position

$$\mathbf{x}(t) = \int_0^t \mathbf{v}(s)\,\mathrm{d}s = V_0 t \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} + \begin{pmatrix} \frac{1}{3}\Omega g t^3 \cos\varphi \\ 0 \\ -\frac{1}{2} g t^2 \end{pmatrix} + \Omega V_0 t^2 \begin{pmatrix} -\cos\varphi \sin\alpha \\ -\sin\varphi\cos\alpha \\ \cos\varphi\cos\alpha \end{pmatrix}.$$

Compared to the parabolic profile, the Coriolis force added all the terms containing Ω. The projectile is shifted to the south (component y is negative). Moreover, the time until the projectile falls down gets longer - if we set the last component of the position equal to zero, we get

$$t = \frac{2V_0 \sin\alpha}{g - 2\Omega V_0 \cos\varphi \cos\alpha}.$$

Problem 2.
Find the Coriolis force acting on an air particle moving in Prague (50°N, 14°E) with the velocity with northward and eastward component, both 10 m/s. Evaluate the size of *all* terms. Does it make sense to omit some terms?

Solution:
On each place on the Earth, we consider local cartesian system, where x axis points to the east, y axis to the north and z axis upward (the same system as in the previous problem). In this system, the angular velocity vector of the Earth rotation is

$$\boldsymbol{\Omega} = (0, \Omega\cos\varphi, \Omega\sin\varphi)\,,$$

where $\Omega = 2\pi/(24\text{ h})$ is the size of the angular velocity and φ is the latitude. We will first consider general velocity vector $\mathbf{u} = (u, v, w)$. Coriolis force on the unit mass can be evaluated as

$$\mathbf{f_{Cor}} = -2\boldsymbol{\Omega} \times \mathbf{u} = \begin{pmatrix} -2\Omega\,(w\cos\varphi - v\sin\varphi) \\ -2\Omega u \sin\varphi \\ 2\Omega u\cos\varphi \end{pmatrix}.$$

Equations of motion are then

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{f_{Cor}}.$$

Now in numbers. The size of the angular velocity in the SI units is approximately $7.3\mathrm{e}^{-5}\ \mathrm{s}^{-1}$. For $\varphi = 50^\circ$, we have $\sin\varphi = 0.77$ and $\cos\varphi = 0.64$. Further, we get $2\Omega\sin\varphi = 1.1\mathrm{e}^{-5}\ \mathrm{s}^{-1}$ and $2\Omega\cos\varphi = 0.93\mathrm{e}^{-5}\ \mathrm{s}^{-1}$. With the velocity $\mathbf{u} = (10, 10, 0)$ m/s, we therefore get

$$\mathbf{f_{Cor}} \approx \begin{pmatrix} 1.1 \\ -1.1 \\ 0.93 \end{pmatrix} 10^{-4}\ \mathrm{m/s}^2.$$

Coriolis force therefore acts in all directions with the similar forces. From their comparison, we therefore cannot say that the effect is negligible in a specific direction. The difference between the terms is however visible from the equations of motion - in the vertical direction, the Coriolis force is very small compared to the gravity. If a specific term is important in the equations depends on the scale in which we study them (e.g., how large is the term $\mathrm{d}\mathbf{u}/\,\mathrm{d}t$).