9. Vorticity

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Problem 1.
Consider the following two examples of incompressible inviscid two-dimensional flow with constant density ρ, described by the velocities

$$\mathbf{u}_1 = (2Ay, -2Ax), \quad \mathbf{u}_2 = (\frac{Ay}{x^2+y^2}, -\frac{Ax}{x^2+y^2})$$

Find the vorticity for both the flows. If the flow is irrotational, find also the pressure, assuming that the gravity is negligible and the pressure for $r \to \infty$ equals a known value p_∞.

Solution:
In the first case, vorticity ($\omega = \partial v/\partial x - \partial u/\partial y$) is

$$\omega_1 = -2A - 2A = -4A.$$

In the second case, it is

$$\omega_2 = \frac{Ax^2 - Ay^2}{x^2+y^2} - \frac{Ax^2 - Ay^2}{x^2+y^2} = 0.$$

The second flow is therefore irrotational.

For the stationary incompressible irrotational flow, we have Bernoulli equation in the form $\rho(u^2 + v^2)/2 + \rho g z + p = \text{const}$.

Neglecting the gravitational force, we therefore get

$$p = C - \frac{1}{2}\rho\frac{A^2}{x^2+y^2},$$

where C is a constant that can be obtained from the condition in infinity. The result is

$$p = p_\infty - \frac{1}{2}\rho\frac{A^2}{x^2+y^2}.$$

Problem 2.
Find the stream functions for the following stream functions. Test whether the velocities correspond to a potential flow and the fields are incompressible.

$$\psi_1 = Axy, \quad \psi_2 = A(x^2 - y^2).$$

Solution:
2D potential flow can be described by the velocity potential, whose gradient defines the velocity

(potential flow means that there exist a potential, which is equivalent to the condition on zero curl). This means that the curves of constant velocity potential connect the points with the same velocity.

The idea of the stream function, another useful function to describe the 2D flow, is to connect points tangential to the velocity vector - to be constant along streamlines ($dx/u = dy/v$). These curves should be orthogonal to curves of constant velocity potential. Hence the definition of the stream function ψ is given by formulas $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$.

In the first case, we therefore have

$$u_1 = Ax, \quad v_1 = -Ay.$$

The zero vorticity condition (potential flow) is satisfied in this case. The divergence is also zero, hence the flow is incompressible.

In the second case, the velocities are

$$u_1 = -2Ay, \quad v_1 = -2Ax.$$

The vorticity and divergence is here also zero.

The reason why the flow defined by the stream function has always zero divergence follows directly from substituting the stream function definition to the incompressibility condition.

The condition on zero vorticity ensures that for these fields also a velocity potential exists. For general fields, it should be always possible to decompose the fields to curl-less and divergence-less part by Helmholtz decomposition and describe it by the combination of a velocity potential and a stream function.

Problem 3.
For the following fields, find the stream function and the velocity potential:

a) Couette flow: Flow between two infinitely long horizontal plates with distance h. One of them is moving with velocity U and the second one is stationary. The velocity of the Couette flow is

$$u = U\frac{y}{h}, \quad v = 0.$$

b) Velocity field

$$u = A(x^2 - y^2), \quad v = -2Axy,$$

where A is a constant.

Solution:
a) Vorticity equals $0 - \mathrm{d}u/\,\mathrm{d}y = -U/h$, divergence is zero. The resulting flow can be therefore described by the stream function ψ, obtained by the integration of equations $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. From the equations:

$$\psi = \frac{1}{2}U\frac{y^2}{h} + f(x), \quad \psi = g(y).$$

Combination of the previous gives

$$\psi = \frac{1}{2}U\frac{y^2}{h} + C,$$

where C is an arbitrary constant from set by fixing one point of the stream function.

b) Vorticity in this case is $-2Ay + 2Ay = 0$, divergence is $2Ax - 2Ax = 0$. This means that both stream function and velocity potential are defined and they are equivalent (it is possible to pass between them). Let us compute again the stream function: By integrating the definition, we would get

$$\psi = A\left(x^2y - \frac{1}{3}y^3\right) + C.$$

Alternatively, we can integrate equations $u = \partial\varphi/\partial x$ and $v = \partial\varphi/\partial y$ to obtain

$$\varphi = -A(xy^2 - \frac{1}{3}x^3) + C.$$

Any of these functions is sufficient to describe the flow.