9. Thermodynamics and waves

12. December 2024

Problem 1.
The atmosphere of Mars is composed mainly of CO_2, which has the specific heat at constant pressure $c_p^m = 844$ J/kg/K and the specific heat at constant volume $c_v^m = 655$ J/kg/K. The gravity on the surface is 3.7 m/s^{-2} and the temperature in the summer near the equator is around 20°C.

- Compute the adiabatic lapse rate and compare it with the lapse rate on the Earth.
- Compute the speed of sound and compare it with the speed of sound on the Earth.

The specific heats for the Earth are $c_p = 1005$ J/kg/K and $c_v = 718$ J/kg/K.

Solution:
The adiabatic lapse rate is defined as

$$\Gamma = -\frac{\mathrm{d}T}{\mathrm{d}z}$$

Using the condition on adiabaticity $c_p \,\mathrm{d}T - \mathrm{d}p/\rho = 0$ with the density ρ and the equation for hydrostatic equilibrium $\mathrm{d}p/\,\mathrm{d}z = -g\rho$, this simplifies to

$$\Gamma = \frac{g}{c_p}.$$

With the conditions on Mars, this is 0.0044 K/m. On the Earth, we get 0.0098 K/m. This means that on the Earth, the temperature decreases by 1 degree on 100 m, in the neutrally stable atmosphere on Mars (in hydrostatic equilibrium), it would be only half of a degree on 100 m due to the different chemical composition and gravity. In reality, the laps rate on Mars is even lower (around 2.5 K/km), because the suspended dust particles absorb solar radiation and heat the air.

As derived on the lecture, the equations of motion imply the formula for the speed of sound $c_s = \sqrt{\gamma R T}$. Both the Poissson constant γ and the specific gas constant R in this equation depend on the chemical composition of the atmosphere. They are defined by $R = c_p - c_v$ and $\gamma = c_p/c_v$. Therefore, the speed of sound is given by

$$c_s = \sqrt{\frac{c_p}{c_v}\left(c_p - c_v\right) T}$$

For the martian atmosphere with the given temperature, we have the speed of sound 267 m/s. For the Earth with the same temperature, it would be 343 m/s.

Problem 2.
Look at a particle moving from east to west on the northern hemisphere, that was dislocated to a

more poleward position by some external perturbation. What will happen then, if we consider the effect of the Coriolis force in the form $\mathbf{F} = (fv, -fu, 0)$, where $f = 2\Omega \sin\varphi$ for the latitude φ? Consider barotropic (density is a function of pressure), horizontal and non-divergent flow, for which the vorticity equation takes the form $\mathrm{d}(\zeta + f)/\,\mathrm{d}t = 0$, where ζ is the relative vorticity.

Solution:
If the particle was moving towards the east, it would describe the motion of the Rossby waves. However, in this case, it is moving in the opposite direction.

First, we notice that the Coriolis parameter f is 0 at the equator and increases towards the pole.

Second, we note that the Coriolis form in this form is perpendicular to the velocity ($\mathbf{F} \cdot \mathbf{v} = 0$) and for positive f (Northern hemisphere) it points to the right from the moving particle (for example, when considering the velocity $(-1, 0, 0)$ – corresponding Coriolis force would be $(0, f, 0)$).

The particle that was displaced on the northern hemisphere towards north was displaced towards the pole, the Coriolis effect will therefore increase. From the vorticity equation, this means that the relative vorticity decreases. This implies that the particle will have larger tendency to rotate in the negative (anticyclonic, clockwise) direction. It will therfore start a circular motion in the direction against the flow. After it goes to the furthest point, the parameter f will however start decreasing and the vorticity would be decreasing, too (it will be still anticyclonic, only smaller).

After some time, it will cross its original trajectory, reach the original latitude and continue with the same motion type. All in all, it will therefore pursue a circular motion. In contrary to the Rossby waves, the circular motion will be in the direction against the direction of motion of the particles.

This motion is called easterly waves or waves in the easterlies. Due to the trade winds, they emerge in the tropical regions as elongated areas of relatively low air pressure (atmospheric troughs) and can even lead to the formation of tropical cyclones in the north Atlantic and northeastern Pacific.

Problem 3.
Equation for geostrophic equilibrium with friction can be written in the form

$$\mathbf{f} \times \mathbf{u} = -\frac{1}{\rho_0} \nabla p + \frac{1}{\rho_0} \frac{\partial \boldsymbol{\tau}}{\partial z}$$

where $\mathbf{f} = (f, -f)$ and $\boldsymbol{\tau}$ is the stress vector. We are considering the Boussinesq aproximation, in which the density is divided into the constant density ρ_0 and a small perturbation.

Over the tropical ocean, we can observe atmospheric cyclons. Compute the effect of such a storm on the Ekman layer in the infinitely deep ocean.

The stress of air caused by a cyclonic storm can be described as

$$\boldsymbol{\tau} = -A \exp^{-(r/\lambda)^2} (y, x, 0),$$

where $r^2 = x^2 + y^2$ and A and λ are constants. The ocean is further affected by the turbulence inside the ocean flow $\boldsymbol{\tau}_t = \rho_0 K \partial_z \mathbf{u}$, where K is a constant. Assume that the flow can be divided into a homogeneous geostrophic component $(\bar{u}, \bar{v}, 0)$ and a perturbation corresponding to the Ekman layer.

Solution:

The equations of motion in components can be written as

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\partial_x p + \frac{1}{\rho_0}\partial_z \tau_x, \\
fu &= -\frac{1}{\rho_0}\partial_y p + \frac{1}{\rho_0}\partial_z \tau_y, \\
0 &= -\frac{1}{\rho_0}\partial_z p + \frac{1}{\rho_0}\partial_z \tau_z.
\end{aligned}
$$

For the geostrophic component, we have equations

$$
\begin{aligned}
-f\bar{v} &= -\frac{1}{\rho_0}\partial_x p \\
f\bar{u} &= -\frac{1}{\rho_0}\partial_y p \\
0 &= -\frac{1}{\rho_0}\partial_z p
\end{aligned}
$$

and for the perturbation part therefore holds

$$
\begin{aligned}
-fv' &= \frac{1}{\rho_0}\partial_z \tau_x, \\
fu' &= \frac{1}{\rho_0}\partial_z \tau_y, \\
0 &= \frac{1}{\rho_0}\partial_z \tau_z.
\end{aligned}
$$

The equations will be solved inside the ocean with the boundary condition given by the stress from the storm. Inside the ocean, we thus have only the stress $\boldsymbol{\tau}_t$ and the equations are therefore

$$
\begin{aligned}
-f(v - \bar{v}) &= K\partial_z^2 u, \\
f(u - \bar{u}) &= K\partial_z^2 v, \\
0 &= K\partial_z^2 w,
\end{aligned}
$$

with the solution satisfying the condition on the ocean surface

$$
z = 0: \qquad \rho_0 K\partial_z(u, v, w) = \boldsymbol{\tau} = -A\exp^{-(r/\lambda)^2}(y, x, 0).
$$

and on the ocean bottom

$$
z = -\infty: \qquad (u, v, w) = (\bar{u}, \bar{v}, 0).
$$

For the solution of the set of equations, we will use the notation $U = u + \mathrm{i}v$, $\bar{U} = \bar{u} + \mathrm{i}\bar{v}$, we multiply the second equation by the imaginary constant and we sum them. By that, the set is converted into a single equation

$$
\mathrm{i}f(U - \bar{U}) = K\partial_z^2 U = K\partial_z^2 (U - \bar{U})
$$

with the boundary conditions $U = \bar{U}$ on the ocean bottom and $\rho_0 K\partial_z U = -A\exp(-(r/\lambda)^2)(y + \mathrm{i}x)$ on the surface.

The solution is thus of the shape

$$
U - \bar{U} = C_1 e^{-\alpha z} + C_2 e^{\alpha z},
$$

with the frequency α satisfying the equation $\alpha^2 = \mathrm{i}f/K$. The De Moivre's formula implies for a complex number $z = \cos\phi + \mathrm{i}\sin\phi$ the relation $z^n = \cos n\phi + \sin n\phi$, and therefore $\alpha = (1+\mathrm{i})\sqrt{f/2K}$. Constants C_1 and C_2 can be determined from the boundary conditions. Because of the condition for the bottom, the constant C_1 equals zero. To get C_2 we differentiate the solution with respect to z and we require

$$\rho_0 K(-\alpha C_1) = -A\exp(-(r/\lambda)^2)(y + \mathrm{i}x).$$

The solution is therefore

$$\begin{aligned}
U &= \bar{U} + \frac{A\sqrt{2K}}{\rho_0 K(1+\mathrm{i})\sqrt{f}}(y+\mathrm{i}x)\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2}\mathrm{e}^{-(1+i)\sqrt{\frac{f}{2K}}z} = \\
&= \bar{U} + \frac{\sqrt{2}A}{\rho_0\sqrt{fK}}\frac{1-i}{2}(y+ix)\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2-\sqrt{\frac{f}{2K}}z}\left(\cos\sqrt{\frac{f}{2K}}z + \mathrm{i}\sin\sqrt{\frac{f}{2K}}z\right) = \\
&= \bar{U} + \frac{A}{\rho_0\sqrt{2fK}}\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2-\sqrt{\frac{f}{2K}}z}(y+x+\mathrm{i}x-\mathrm{i}y)\left(\cos\sqrt{\frac{f}{2K}}z + \mathrm{i}\sin\sqrt{\frac{f}{2K}}z\right) = \\
&= \bar{U} + \frac{A}{\rho_0\sqrt{2fK}}\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2-\sqrt{\frac{f}{2K}}z} \\
&\left[(x+y)\cos\sqrt{\frac{f}{2K}}z + (y-x)\sin\sqrt{\frac{f}{2K}}z + \mathrm{i}\left((x+y)\sin\sqrt{\frac{f}{2K}}z + (x-y)\cos\sqrt{\frac{f}{2K}}z\right)\right].
\end{aligned}$$

After dividing into the real and the complex component, we get

$$u = \bar{u} + \frac{A}{\rho_0\sqrt{2fK}}\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2-\sqrt{\frac{f}{2K}}z}\left[(x+y)\cos\sqrt{\frac{f}{2K}}z + (y-x)\sin\sqrt{\frac{f}{2K}}z\right],$$

$$v = \bar{v} + \frac{A}{\rho_0\sqrt{2fK}}\mathrm{e}^{-\left(\frac{r}{\lambda}\right)^2-\sqrt{\frac{f}{2K}}z}\left[(x+y)\sin\sqrt{\frac{f}{2K}}z + (x-y)\cos\sqrt{\frac{f}{2K}}z\right].$$

The expression $\sqrt{2K/f}$ corresponds to the depth of the Ekman layer. The exponential with the expression r/λ causes the attenuation of the perturbation in the direction from the storm.