10. Coriolis force 2

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Problem 1.
Consider air moving toward the north with the velocity $v = 10$ m/s at 45°N. How much will it deviate eastward within an hour?

Solution:
In the standard local coordinate system, the angular velocity vector of the Earth rotation is

$$\mathbf{\Omega} = (0, \Omega \cos \varphi, \Omega \sin \varphi)\,,$$

where $\Omega = 2\pi/(24 \text{ h})$ is the size of the angular velocity and φ is the latitude. For the velocity vector $\mathbf{u} = (u, v, w)$, Coriolis force on the unit mass can be evaluated as

$$\mathbf{f_{Cor}} = -2\mathbf{\Omega} \times \mathbf{u} = \begin{pmatrix} -2\Omega\left(w \cos \varphi - v \sin \varphi\right) \\ -2\Omega u \sin \varphi \\ 2\Omega u \cos \varphi \end{pmatrix}.$$

The eastward component of the force is therefore $2\Omega v \sin \varphi$. If we consider this force only, we can write

$$\frac{\mathrm{D}u}{\mathrm{D}t} = 2\Omega v \sin \varphi$$

The derivative on the left-hand side is the material derivative

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u$$

However, if we consider again only the constant northward wind component as the most significant background wind (not being changed by the Coriolis force), we can neglect the ∇u term and approximate the derivative as the partial derivative.

We therefore have

$$\frac{\partial u}{\partial t} \approx 2\Omega v \sin \varphi$$

and we can approximate

$$\Delta u \approx 2\Omega v \sin \varphi \Delta t \approx 2 \cdot 7.3 \cdot 10^{-5} \cdot 10 \cdot \sin 45 \cdot 3600 \approx 3.7\,\mathrm{m/s}.$$

This is the end velocity that the particle would have at the end of the hour. Since the velocity was 0 at the beginning, we will consider further only half of the maximum velocity. The change of position can thus be very roughly approximated as

$$\Delta x \approx \frac{\Delta u}{2} \Delta t = \frac{3.7}{2} 3600 = 6.7\,\mathrm{km}$$

However, this change is not negligible and we were doing quite strong assumptions by considering no effects from the induced meridional velocity on the zonal velocity. If we work more carefully, we can get to the so-called inertial oscillation.

Problem 2.
Inertial oscillation is a special type of motion of the air in the atmosphere, in which the inertia of the fluid is balanced by the Coriolis force.
a) This fluid can be described in the eulerian description by equations

$$\frac{\partial u}{\partial t} - fv = 0,$$
$$\frac{\partial v}{\partial t} + fu = 0,$$

where f is the Coriolis parameter, taken as a constant here. Find the period of the oscillations.
b) In the lagrangean description, one can write (outside the equatorial region) equations for motion of an air parcel

$$-K_H|\mathbf{u}|\mathbf{u} \times \mathbf{k} = f\mathbf{u} \times \mathbf{k},$$

where K_H is the horizontal curvature of the motion and $\mathbf{k}$ is a vector pointing upwards in the direction of the z-axis. The equation describes the equality of the centrifugal force created due to the horizontal curvature of the streamlines and the Coriolis force. How does this motion look like in the middle latitudes ($f \approx 10^{-4}$ s^{-1}) with the flow velocity 10 m/s, if the Coriolis parameter is constant? How does the trajectory change, if we consider the dependence of the Coriolis parameter on the latitude $f = 2\Omega \sin(\varphi)$ (Ω is the angular frequency for the rotation of the Earth)?

Solution:
a) In the first step, we will solve the set of differential equations for u and v. One can either guess the solution (insert an einsatz in the equations) or there is a trick:

We define $V = u + \mathrm{i}v$, where i is an imaginary constant. The set of equation can be than converted to

$$\frac{\partial V}{\partial t} + \mathrm{i}fV = 0,$$

with the solution $V = V_0 e^{-ift}$. From this, we see that the particles are moving with the period $2\pi/f$. A better insight can be however obtain from the lagrangian description in the part b).

b) If the value of the Coriolis parameter f would not depend on the latitude, the air parcel in the equilibrium state would be moving in circle. The Coriolis vector would point to the centre of the circle and the centrifugal force vector with the same size would head to the opposite direction.

On the northern hemisphere, the Coriolis force is always pointing to the right with respect to the direction of motion. The particle therefore moves in the circle in the clockwise direction. On the southern hemisphere, the Coriolis force is pointing to the left, so the particle moves counterclockwise. (On both hemispheres, this is anticyclonal direction - the direction in which motion moves around the pressure high.)

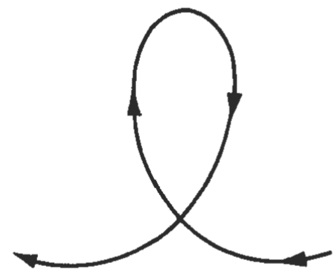
From the equation, we can find other properties of the inertial circulation:

For the size of the velocity vector u_{in}, we have

$$u_{in} = -\frac{f}{K_H} = -fR_{in},$$

where R_{in} is the radius of the inertial circle ($R_{in} < 0$ on the northern hemisphere). For example, if the velocity of the circulation in midlatitudes ($f \approx 10^{-4}$ s^{-1}) is 10 m/s, the radius of the circle is approximately 100 km.

(a) Inertial circulation on the earth.
Taken from https://commons.wikimedia.org/wiki/File:Coriolis_effect14.png#/media/File:Coriolis_effect14.png



(b) The shape of the inertial circculations with the latitudinal changes of the Coriolis parameter. Taken from the book Příručka dynamické meteorologie, Pechala, Bednář

Obrázek 1: Inerční cirkulace.

The time to complete one orbit (=inertial period) is given as

$$T_{in} = \frac{2\pi|R_{in|}}{u_{in}} = \frac{2\pi}{|f|}.$$

For the middle latitudes, the resulting inertial period is approximately 17 hours.

Because f grows in the direction from the equator to poles, for the constant velocity u_{in}, the radius of the oscillation at poles is smaller than near equator. Schematic figure of such oscillations is depicted in Fig. 1a.

In reality, the Coriolis parameter is however varying also during the circulation and the particles are therefore not moving in circles. The curvature is higher further from the equator than at the equator – the resulting trajectory for the northern hemisphere is depicted in Fig. 1b.

Problem 3.
Stress in Newtonian (Navier-Stokes) equation can be described by the formula

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D},$$

where $\mathbb{D}$ is the symmetric part of the velocity gradient defined as $\mathbb{D} = (D_{ij})_{i,j=1,2,3}$, $D_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$. Show that for the simple shear flow $\mathbf{u} = (u(y), 0, 0)$, all diagonal components of the stress tensor $\mathbb{T}$ are the same. Consider further the Stokes fluid, for which we have (instead of the previous formula)

$$\mathbb{T} = -p\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2.$$

How does the tensor $\mathbb{T}$ look like this time? Can this have some consequences for the flow? (The equations of motion in this formulation are written as $\rho\, \mathrm{d}\mathbf{u}/\,\mathrm{d}t = \nabla \cdot \mathbb{T} + \rho\mathbf{b}$, where $\mathbf{b}$ is an external force.)

Solution:
For the simple shear flow, we have

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the newtonian fluid, it therefore holds

$$\mathbb{T} = \begin{pmatrix} -p & \mu u' & 0 \\ \mu u' & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

whereas for the Stokes fluid, we get

$$\mathbb{T} = \begin{pmatrix} -p + \frac{\alpha_2}{4}u'^2 & \frac{\alpha_1}{2}u' & 0 \\ \frac{\alpha_1}{2}u' & -p + \frac{\alpha_2}{4}u'^2 & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

The diagonal components in the Stokes fluid are therefore not the same.

From the practical point of view, this is an important difference between the newtonian fluid and some nonnewtonian fluids. There is some stress in the direction perpendicular to the direction of flow. This results in some effects, such as the Weissenberg effect (the fluid is climbing upward on a rotating rod), Barus effect (when the fluid moves out of a narrow space, e.g. the water tap, it widens more than the Newtonian fluid), fluid moving down on an inclined plane creates higher layer at the front, the pressure measured in the hole of a pipe does not correspond to the pressure inside the pipe etc.