(P1) ! není třeba řešit $(\Omega,\mathcal{A},P)$. Hledáme P_X od diskr. náh. vel. Stačí ji určit z $P(X\in B), B\in\mathcal{B}$

a) zřejmě $P(X\in\{0,1,2\})=1$ ... X nabývá hodnot z $\{0,1,2\}$ ... má diskr. rozdělení

§3

$$P(X=0) = P(X=0|T_1)\cdot P(T_1)+P(X=0|T_2)P(T_2) = \frac{1}{\binom{4}{2}}\cdot\frac{1}{2}+\frac{1}{\binom{3}{2}}\cdot\frac{1}{2} = \frac{1}{2}\left(\frac{1}{6}+\frac{1}{3}\right)=\frac{1}{4}$$

vo úplné psti – předpoklady:

$T_1=\{$vybereme 1. truhlu$\}$ $\quad T_1\cap T_2=\emptyset \quad P(T_1)=P(T_2)=\frac{1}{2}>0 \quad P(T_1\cup T_2)=1$ OK

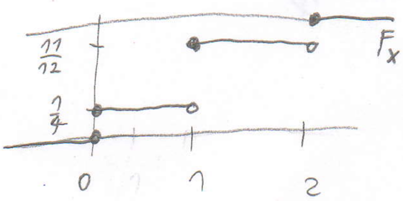
$T_2=\{$ — " — 2. truhlu$\}$

$$P(X=1) = P(X=1|T_1)P(T_1)+P(X=1|T_2)P(T_2) = \frac{\binom{2}{1}^2}{\binom{4}{2}}\cdot\frac{1}{2}+\frac{1\cdot\binom{2}{1}}{\binom{3}{2}}\cdot\frac{1}{2} = \frac{1}{2}\left(\frac{4}{6}+\frac{2}{3}\right)=\frac{2}{3}$$

$$P(X=2) = P(X=2|T_1)P(T_1)+P(X=2|T_2)P(T_2) = \frac{1}{\binom{4}{2}}\cdot\frac{1}{2}+0\cdot\frac{1}{2}=\frac{1}{12}$$ a nebo dopočet do 1

tedy X má diskr. rozd. s nosičem $S=\{0,1,2\}$ a pstmi $\left\{\frac{1}{4},\frac{2}{3},\frac{1}{12}\right\}$

b) $P_X=\frac{1}{4}\delta_0+\frac{2}{3}\delta_1+\frac{1}{12}\delta_2 \qquad F_X(x)=\frac{1}{4}\mathbb{1}_{[0,\infty)}(x)+\frac{2}{3}\mathbb{1}_{[1,\infty)}(x)+\frac{1}{12}\mathbb{1}_{[2,\infty)}(x)$



c) $$EX=\int_{\mathbb{R}} x\,dP_X(x)=\sum_{i=0}^{2}x_i p_i=\sum_{i=0}^{2} i\,P(X=i)=1\cdot\frac{2}{3}+2\cdot\frac{1}{12}=\frac{4+1}{6}=\frac{5}{6}$$

$$\operatorname{var}X=EX^2-(EX)^2=\sum_{i=0}^{2} i^2 P(X=i)=1\cdot\frac{2}{3}+4\cdot\frac{1}{12}-\frac{25}{36}=\frac{11}{36}$$

d) $Y(\omega)=X(\omega)\cdot 100 \quad \forall\omega\in\Omega$

zřejmě Y diskr. náh. vel nabývá hodnot z $S_Y=\{0,100,200\}$ s pstmi $\left\{\frac{1}{4},\frac{1}{3},\frac{1}{12}\right\}$

neboť $P(Y=100)=P(X=1)=\frac{2}{3}$

$P(Y=0)=P(X=0)=\frac{1}{4}$ $\qquad P_Y=\frac{1}{4}\delta_0+\frac{2}{3}\delta_{100}+\frac{1}{12}\delta_{200}$

$P(Y=200)=P(X=2)=\frac{1}{12}$

$EY=E\,100.X=100EX=\frac{500}{6}$

$\operatorname{var}Y=\operatorname{var}(100X)=100^2\operatorname{var}X=\frac{1100}{36}$

[snadné a rychlé počítat takto:
$EY=Eh(X)$
$\operatorname{var}Y=\operatorname{var}h(Y)$

(P2) $X:\Omega\to\{1,2\} \quad X^{-1}\{1\}=\{1,2\}\in\mathcal{A} \quad X^{-1}\{2\}=\{3,4\}\in\mathcal{A} \quad (\{1\},\{2\})$ generátor $\mathcal{P}(\{1\},\{2\})$

tedy X měřitelná $\Rightarrow X$ n.v.

$Y:\Omega\to[1,2] \quad Y^{-1}\{1\}=\{1,2,3\}\notin\mathcal{A}$ tedy Y není měřitelná, není n.v.

P3 $X: \Omega \to \mathbb{R}$ $\quad p \in (0,1)$ $\quad$ vidíme, co je P_X? X? $\quad X \sim Alt(p)$

a) $X: \Omega \to \{0,1\}$ diskrétní

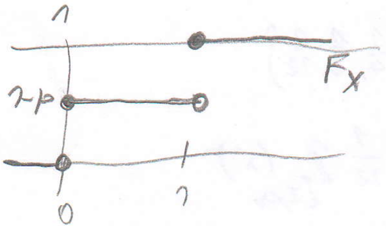
$P(X=1) = P(\{\omega : I(\omega < p)\}) = \int_0^1 I(\omega < p)\, d\omega = p$ $\quad \Rightarrow P_X = p\delta_1 + (1-p)\delta_0$

$P(X=0) = P(\{\omega : I(\omega \geq p)\}) = \int_p^1 d\omega = (1-p)$ $\quad$ ano, alt. rozd

$$F_X(x) = P(X \leq x) = P(\{\omega : X(\omega) \leq x\}) = P(\{\omega : 1_{(0,p]}(\omega) \leq x\}) = \begin{cases} 0 & x<0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

anebo

$$F_X(x) = (1-p)\, 1_{[0,\infty)}(x) + p\, 1_{[1,\infty)}(x) \quad \sim P_X$$



skoky v bodech S_X ... nosiče P_X

b) $X \sim Alt(p) \Rightarrow EX = 0 \cdot (1-p) + 1 \cdot p = \underline{p}$

$\quad var X = EX^2 - (EX)^2 = 0\cdot(1-p) + 1 \cdot p - p^2 = \underline{p(1-p)}$

Takže: pokud máme $(\Omega, \mathcal{A}, P)$ a X jako funkci, vše je snadné. Prostě určím P_X.

P4 a) můžeme např. $(\Omega, \mathcal{A}, P) = (\{1,2,3,4\}^n, \mathcal{P}(\Omega), P)$, $P(\omega) = \frac{1}{4^n}$, $\omega \in \Omega$

búno správná 1, pak $X(\omega) = \#1 = \sum_{i=1}^n 1_{\{1\}}(\omega_i)$ $\quad \omega = (\omega_1, \dots, \omega_n)$

$0 \leq k \leq n$

$$P(X=k) = \frac{1}{4^n} \sum_{\omega : \sum_1^n 1_1(\omega_i) = k} 1 = \frac{1}{4^n}\binom{n}{k} 3^{n-k} = \binom{n}{k}\left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k}$$

binomické rozdělení s param (n,k)

$$\left(\text{příp. } P_X = \sum_{k=0}^n \binom{n}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{n-k} \delta_k\right)$$

b) $$EX = \sum_{k=0}^n k \binom{n}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{n-k} = \frac{n}{4} \underbrace{\sum_{k=1}^n \binom{n-1}{k-1}\left(\frac{1}{4}\right)^{k-1}\left(\frac{3}{4}\right)^{n-k}}_{=\left(\frac{1}{4}+\frac{3}{4}\right)^n = 1} = \underline{\underline{\frac{n}{4}}}$$

(anebo vím $EX = np = \underline{\underline{\frac{n}{4}}}$)

c) $var X = EX^2 - (EX)^2$

$$EX(X-1) = \sum_{k=2}^n k(k-1)\binom{n}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{n-k} = \frac{n(n-1)}{4 \cdot 4} \underbrace{\sum_{k=2}^n \binom{n-2}{k-2}\left(\frac{1}{4}\right)^{k-2}\left(\frac{3}{4}\right)^{n-k}}_{1} = \frac{n(n-1)}{4 \cdot 4}$$

$$EX^2 = EX(X-1) + EX = \frac{n(n-1)}{4\, 4} + \frac{n}{4} = \frac{n^2+3n}{4 \cdot 4}$$

$$var X = EX^2 - (EX)^2 = \frac{n^2 + 3n - n^2}{4 \cdot 4} = \frac{3n}{16} \quad (= npq)$$

vímo binom rozd

cv3

(P4) d) $Y(\omega) = X(\omega)\cdot a - (n - X(\omega))$ chci $EY = 0$

$EY = E(aX - (n-X)) = aEX - n + EX = (1+a)EX - n$

$(1+a)\frac{n}{4} - n = 0$

$\frac{(1+a)}{4} = 1 \Rightarrow \underline{\underline{a = 3}}$

e) $EX = \frac{n}{k}$ $\quad \operatorname{var} X = \frac{n}{k}\left(1 - \frac{1}{k}\right) = \frac{n(k-1)}{k^2}$

najděme maximum fce $h(k) = \frac{k-1}{k^2}$

$h'(k) = \left(\frac{k-1}{k^2}\right)' = \frac{k^2 - (k-1)2k}{k^4} = \frac{k^2 - 2k^2 + 2k}{k^4} = \frac{2k - k^2}{k^4}$

$h'(k) = 0$ $\quad k = 0$ $\quad \operatorname{var} X = 0$

$\underline{k = 2}$ $\quad \operatorname{var} X = \frac{n}{4}$ maximální rozptyl

$h'(k) < 0$ $\quad k > 2$

$h'(k) > 0$ $\quad 0 < k < 2$

(P5) a) $\sum_{k=0}^{\infty} \lambda^k \frac{e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$ je to pstní rozdělení

$EX = \sum_{k=0}^{\infty} k \lambda^k \frac{e^{-\lambda}}{k!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \lambda^{(k-1)} \cdot \frac{1}{(k-1)!} = \lambda e^{-\lambda} \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = \lambda e^{-\lambda} e^{\lambda} = \underline{\underline{\lambda}}$

Poissonovo rozdělení

b) $\operatorname{var} X = EX^2 - (EX)^2 = EX(X-1) + EX - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$

$\underline{EX(X-1)} = \sum_{k=1}^{\infty} k(k-1)\lambda^k \frac{e^{-\lambda}}{k!} = \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = \underline{\lambda^2}$

výhodné použít

(P6) Nemusím řešit $(\Omega, \mathcal{A}, P)$, stačí najít $P(X=k)$, $k \in \mathbb{N}_0$ z popisu situace.

$P(X=k) = (1-p)^k p$ $\quad k \in \mathbb{N}_0$ neb nezávislé pokusy

$\sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} (1-p)^k p = p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1$ OK pstní rozdělení! ← dobré ověřit

$EX = \sum_{k=0}^{\infty} k P(X=k) = \sum_{k=1}^{\infty} k p (1-p)^k = p(1-p) \sum_{k=1}^{\infty} k (1-p)^{k-1} = \frac{p(1-p)}{(1-(1-p))^2} = \underline{\underline{\frac{1-p}{p}}}$

geometrické rozdělení