Constitutive Equations

$$\boldsymbol{S} = \nu(e)\boldsymbol{D}(\boldsymbol{v}) \qquad 2\boldsymbol{D}(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T$$
$$\boldsymbol{q} = \kappa(e)\nabla e$$

Governing equations

$$\operatorname{div} \boldsymbol{v} = 0$$
$$\boldsymbol{v}_t + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \operatorname{div}\Big(\nu(e)\boldsymbol{D}(\boldsymbol{v})\Big)$$
$$(e + \tfrac{1}{2}|\boldsymbol{v}|^2)_t + \operatorname{div}((e + \tfrac{1}{2}|\boldsymbol{v}|^2)\boldsymbol{v}) - \operatorname{div}(\kappa(e)\nabla e) = \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}(\nu(e)\boldsymbol{D}(\boldsymbol{v})\,\boldsymbol{v})$$

and

$$e_t + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}(\kappa(e)\nabla e) \geq \nu(e)|\boldsymbol{D}(\boldsymbol{v})|^2$$

Weak solution (BM, BLM, BE) vrs **Suitable weak solution** (BM, BLM, BE + Entropy inequality):

2D, 3D

Energy estimates and their consequences

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \boldsymbol{v}_t + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \operatorname{div}\Big(\nu(\dots)\boldsymbol{D}(\boldsymbol{v})\Big)$$

$$\left(e + \frac{|\boldsymbol{v}|^2}{2}\right)_t + \operatorname{div}\left(\left(e + \frac{|\boldsymbol{v}|^2}{2}\right)\boldsymbol{v}\right) - \operatorname{div}(\kappa(\dots)\nabla e) = \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}\Big(\nu(\dots)\boldsymbol{D}(\boldsymbol{v})\,\boldsymbol{v}\Big)$$

$$e_t + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}(\kappa(\dots)\nabla e)(\geq) = \nu(\dots)|\boldsymbol{D}(\boldsymbol{v})|^2$$

- $\int_\Omega \left(e + \frac{|\boldsymbol{v}|^2}{2}\right)(t,x)dx \leq \int_\Omega \left(e_0 + \frac{|\boldsymbol{v}_0|^2}{2}\right)dx \quad \Longrightarrow \quad \boxed{e \in L^\infty(L^1) \quad \boldsymbol{v} \in L^\infty(L^2)}$
- $\int_0^T \nu(\dots)|\boldsymbol{D}(\boldsymbol{v})|^2\,dx \leq C \quad \Longrightarrow \quad \boxed{\nabla \boldsymbol{v} \in L^2(L^2)}$
- $\nu(\dots)|\boldsymbol{D}(\boldsymbol{v})|^2 \geq 0, \quad \Longrightarrow \quad \boxed{e > C^* \text{ a.e. } , e \in L^m(L^m)\,,\ \nabla(e)^{(1-\lambda)/2} \in L^2(L^2)}$

Equation for the pressure (Navier's slip) $\quad \boxed{p = (-\Delta)^{-1}\operatorname{div}\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v} - \nu(\dots)\boldsymbol{D}(\boldsymbol{v}))}$

- $\boldsymbol{v} \in L^\infty(L^2)$ and $\nabla \boldsymbol{v} \in L^2(L^2) \quad \Longrightarrow \quad \boxed{\boldsymbol{v} \in L^{10/3}(L^{10/3}) \quad \text{and} \quad p \in L^{5/3}(L^{5/3})}$

Energy estimates and their consequences

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \boldsymbol{v}_t + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \operatorname{div}\Big(\nu(\dots)\boldsymbol{D}(\boldsymbol{v})\Big)$$

$$\Big(e + \frac{|\boldsymbol{v}|^2}{2}\Big)_t + \operatorname{div}\Big(\Big(e + \frac{|\boldsymbol{v}|^2}{2}\Big)\boldsymbol{v}\Big) - \operatorname{div}(\kappa(\dots)\nabla e) = \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}\Big(\nu(\dots)\boldsymbol{D}(\boldsymbol{v})\,\boldsymbol{v}\Big)$$

$$e_t + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}(\kappa(\dots)\nabla e) \qquad (\geq) = \qquad \nu(\dots)|\boldsymbol{D}(\boldsymbol{v})|^2$$

- $\boldsymbol{v}_t \in \Big(L^{5/2}(W^{1,5/2})\Big)^* = L^{-5/3}(W^{1,-5/3})$
- $e_t \in L^1(W^{-1,q'})$ with $q > 10$
- Aubin-Lions lemma and its generalization: $\boldsymbol{v}$ and e precompact in $L^m(L^m)$ for $m \in [1, \frac{5}{3})$
- Trace theorem and Aubin-Lions lemma: pre-compactness of $\boldsymbol{v}$ on $\partial\Omega$
- Vitali's theorem

Two steps in the proof of existence

- Stability of the system w.r.t. weakly converging sequences
- Constructions of approximations (several levels), derivation of uniform estimates, weak limits - candidates for the solutions, taking limits in nonlinearities

Result #1

Theorem 1. (**M. Bulíček, E. Feireisl**, J. Málek, '06-'07) Assume that

$$\nu_1 \geq \nu(s) \geq \nu_0 > 0 \text{ and } \kappa_1 \geq \kappa(s) \geq \kappa_0 > 0 \text{ for all } s \in \mathbb{R}$$

Let $\partial\Omega \in C^{1,1}$, $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},div}$ and $e_0 \in L^1$, $e_0 \geq C^* > 0$ a.a. in Ω. Let $g \in L^1(0,T)$.

Then for all $T > 0$ (and any $\alpha \in [0,\infty)$) there exists (suitable) weak solution $(\boldsymbol{v}, p)$ to the system in consideration, completed by Navier's slip boundary conditions, such that

$$\boldsymbol{v} \in C(0,T;\dot{L}^2_{weak}) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},div})$$

$$\mathrm{tr}\,\boldsymbol{v} \in L^2(0,T;L^2(\partial\Omega))$$

$$p \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}) \qquad \int_\Omega p(t,x)dx = g(t)$$

$$e \in L^\infty(0,T;L^1) \cap L^m(0,T;L^m) \cap L^n(0,T;W^{1,n}) \qquad m \in [1,\frac{5}{3}), n \in [1,\frac{5}{4})$$

$$\boxed{(p + \frac{|\boldsymbol{v}|^2}{2})\boldsymbol{v} \in L^{\frac{10}{9}}(0,T;L^{\frac{10}{9}}) \qquad \boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} \in L^{\frac{5}{4}}([0,T];L^{\frac{5}{4}})}$$

Fluids with shear rate dependent viscosities

$$\boxed{\boldsymbol{S} = \nu(|\boldsymbol{D}|^2)\boldsymbol{D}(\boldsymbol{v})}$$

If $\boldsymbol{v} = (u(x_2), 0, 0)$, then $|\boldsymbol{D}(\boldsymbol{v})|^2 = 1/2|u'|^2$ ... shear rate.

- $\nu(|\boldsymbol{D}|^2) = |\boldsymbol{D}|^{r-2}$ $\quad 1 < r < \infty$
- power-law model
- $\nu(|\boldsymbol{D}|^2) \searrow$ as $|\boldsymbol{D}|^2 \nearrow$
- shear thinning fluid ($r < 2$)

- $\nu(|\boldsymbol{D}|^2) = \nu_0 + \nu_1|\boldsymbol{D}|^{r-2}$ $\quad r > 2$
- Ladyzhenskaya model (65)
- (Smagorinskii turbulence model: $r = 3$)

(A) given $\boxed{r \in (1, \infty)}$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$

$$C_1(K + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \le \frac{\partial\left[(\nu(|\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \le C_2(K + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2$$

K can be even 0 in many cases.

Four approaches used in the analysis

- Higher regularity method $\Longrightarrow$ regularity for $\boxed{r \geq 2 + \frac{1}{5}}$, but gives existence for $\boxed{r > 2 - \frac{1}{5}}$

- Monotone operator theory - Test by $\boldsymbol{u}^n - \boldsymbol{u}$

$$v_k^n \partial_k \boldsymbol{v}^n (\boldsymbol{v}^n - \boldsymbol{v}) \in L^1(Q) \iff \boxed{r \geq 2 + \frac{1}{5}}$$

O.A. Ladyzhenskaya '65, J.L. Lions '69

- L^∞ - truncation of Sobolev functions - Test by $(\boldsymbol{v}^n - \boldsymbol{v})(1 - \min(1, \frac{|\boldsymbol{v}^n - \boldsymbol{v}|}{\lambda}))$

$$v_k^n \partial_k \boldsymbol{v}^n \in L^1(Q) \iff \boxed{r > 2 - \frac{2}{5}}$$

J. Frehse, J. Málek, **M. Steinhauer** '00, Wolf '07

- $W^{1,\infty}$ - truncation of Sobolev functions - Test by $(\boldsymbol{v}^n - \boldsymbol{v})_\lambda$

$$v^n \otimes \boldsymbol{v}^n \in L^1(Q) \iff \boxed{r \geq 2 - \frac{4}{5}}$$

Conjecture based on **J. Frehse**, J. Málek, **M. Steinhauer** '03

L. Diening, M. Růžička, J. Wölf '07

Lemma on Lipschitz approximations of Sobolev functions

Lemma for one function: Let Ω smooth, bounded and $u \in W_0^{1,1}(\Omega)$.
Then for every $\lambda > 0$, $\theta > 0$ there is $u_{\theta,\lambda} \in W_0^{1,\infty}(\Omega)$:

- $\|u_{\theta,\lambda}\|_\infty \leq \theta$,
- $\|\nabla u_{\theta,\lambda}\|_\infty \leq c\lambda$,
- $\{u \neq u_{\theta,\lambda}\} \subset \Omega \cap (\{M(u) > \theta\} \cup \{M(|\nabla u|) > \lambda)\}$

Lemma (*Diening, Málek, Steinhauer*) Let $\Omega \in \mathcal{C}^{0,1}$ and $u^n \rightarrow 0$ in $W_0^{1,r}(\Omega)$.
Denote $K := \sup_n \|u^n\|_{1,r}$ and $\gamma_n := \|u^n\|_r \rightarrow 0$ and $\mu_j := 2^{2^j}$. Set $\theta_n := \sqrt{\gamma_n}$.

Then there are $\lambda_{n,j} \in [\mu_j, \mu_{j+1}]$

- $\|u^{n,j}\|_\infty \leq \theta_n$ and $\|\nabla u^{n,j}\|_\infty \leq C\lambda_{n,j}$
- $u^{n,j} \rightarrow 0$ strongly in $L^\infty(\Omega)$
- $u^{n,j} \rightharpoonup 0$ weakly in $W_0^{1,s}(\Omega) \qquad s \in \langle 1, \infty)$

Evenmore, for all n, j

- $\|\nabla u^{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_r \leq c\|\lambda_{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_r \leq c\frac{\gamma_n}{\theta_n}\mu_{j+1} + cK\frac{1}{2^{j/r}}$

Fluids with pressure dependent viscosities

$$\boldsymbol{S} = \nu(p)\boldsymbol{D}(\boldsymbol{v})$$

$\nu(p) = \exp(\gamma p)$

Bridgman(31): "The physics of high pressure"

Cutler, McMickle, Webb and Schiessler(58)

Johnson, Cameron(67), Johnson, Greenwood(77), Johnson, Tewaarwerk(80)

Paluch et al. (99), Bendler et al. (01)

elastohydrodynamics: Szeri(98) synovial fluids

No global existence result.

- Renardy(86), local, ($\frac{\nu(p)}{p} \to 0$ as $p \to \infty$)
- Gazzola(97), Gazzola, Secchi(98): local, severe restrictions

$$\boxed{\partial_t \boldsymbol{v} + \mathbb{P}\,\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \mathbb{P}\,\mathrm{div}(\nu(p)\boldsymbol{D}(\boldsymbol{v})) = 0}$$

$$\boxed{p = (-\Delta)^{-1}\,\mathrm{div}\,\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{v} - \nu(p)\boldsymbol{D}(\boldsymbol{v}))}$$

$\mathcal{F} := (-\Delta)^{-1}\,\mathrm{div}\,\mathrm{div}$ (a Fourier multiplier)

Minimal requirement: $\boldsymbol{v} \mapsto p = p(\boldsymbol{v})$ is well defined.Let p_1, p_2 be two solutions corresponding to $\boldsymbol{v}$.

$$p_1 - p_2 = \mathcal{F}((\nu(p_2) - \nu(p_1))\boldsymbol{D}(\boldsymbol{v})) = \mathcal{F}(\partial_p\nu(p_2 + \theta_1(p_2 - p_1))\,\boldsymbol{D}(\boldsymbol{v})\,(p_2 - p_1))$$

Not clear which side contains the leading operator. A very complex relation.

$$\boxed{\nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)}$$

$$\begin{aligned} p_1 - p_2 &= -\mathcal{F}\Big(\big(\nu(p_1, |\boldsymbol{D}(\boldsymbol{v})|^2) - \nu(p_2, |\boldsymbol{D}(\boldsymbol{v})|^2)\big)\,\boldsymbol{D}(\boldsymbol{v})\Big) \\ &= \mathcal{F}(\partial_p\nu(p_2 + \theta_1(p_2 - p_1), |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v})\,(p_2 - p_1)) \end{aligned}$$

$$\boxed{\begin{aligned} \|p^1 - p^2\|_q &\le \|\partial_p\nu(p^2 + \theta_1(p^2 - p^1), |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v})\,(p^1 - p^2)\|_q \\ &\le \sup_{p,\boldsymbol{D}} |\partial_p\nu(p, |\boldsymbol{D}|^2)\,\boldsymbol{D}|\;\|p^1 - p^2\|_q \end{aligned}}$$

$\nu(p, |\boldsymbol{D}|^2) = \ln(1 + |p| + |\boldsymbol{D}|)$

Fluids with shear rate and pressure dependent viscosities

$$\boxed{\boldsymbol{S} = \nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}(\boldsymbol{v})}$$

$$\nu(p, |\boldsymbol{D}|^2) = (\eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + \delta|\boldsymbol{D}|^{2-r}}) \exp(\gamma\, p) \qquad r = 1.56$$

Davies and Li(94), Gwynllyw, Davies and Phillips(96)

$$\nu(p, |\boldsymbol{D}|^2) = c_0 \frac{p}{|\boldsymbol{D}|} \qquad r = 1 \qquad \text{Schaeffer(87) - instabilities in granular materials}$$

$$\nu(p, |\boldsymbol{D}|^2) = (A + (1 + \exp(\alpha\, p))^{-q} + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}$$

$$\alpha > 0, A > 0 \qquad \boxed{1 \leq r < 2} \qquad 0 \leq q \leq \frac{1}{2\alpha}\frac{r-1}{2-r}A^{(2-r)/2}$$

elastohydrodynamics, synovial fluids, film flows, granular materials

Assumptions on $\boxed{\boldsymbol{S} = \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})}$

(A1) given $\boxed{r \in (1,2)}$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$ and all p

$$C_1(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \leq \frac{\partial\left[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2$$

(A2) for all symmetric matrices $\boldsymbol{D}$ and all p

$$\left|\frac{\partial[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial p}\right| \leq \gamma_0(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{4}} \leq \gamma_0 \qquad \gamma_0 < \frac{1}{C_{div,2}}\,\frac{C_1}{C_1 + C_2}\,.$$

Examples of ν's fulfilling (A1) and (A2)

Consider

$$\nu_i(p, |\boldsymbol{D}|^2) = (\mu_i(p) + |\boldsymbol{D}|^2)^{\frac{r-2}{2}} \qquad i = 1, 2, 3$$

$$\mu_1(p) = A + (1 + \alpha^2 p^2)^{\frac{-q}{2}}$$

$$\mu_2(p) = A + (1 + \exp(\alpha p))^{-q}$$

$$\mu_3(p) = \begin{cases} A + \exp(-\alpha q\, P)) & \text{if } p > 0\,, \\ A + 1 & \text{if } p \leq 0\,. \end{cases}$$

with

$$\alpha > 0, \quad A > 0, \quad q > 0, r \in (9/5, 2) \text{ and } \boxed{\alpha |q| (2 - r) \leq \frac{r-1}{4}}$$

$\boxed{\nu_i(\cdot, |\boldsymbol{D}|^2) \text{ is increasing in the first variable for any fixed } \boldsymbol{D}}$

These models are pressure thickening and shear thinning which is in agreement with experimental observations.

These models fulfil the assumptions **(A1)–(A2)**.

Result #2

Theorem 2. (**M. Bulíček**, J. Málek, **K. R. Rajagopal** '07) Let **(A1)–(A2)** hold and r in **(A1)** satisfy

$$\boxed{r \in \left(\frac{8}{5}, 2\right)}$$

Assume that

- $\partial\Omega \in C^{1,1}$
- $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},div}$
- $g \in L^1(0,T)$

Then for all $T > 0$ (and any $\alpha \in (0,1]$) there exists at least one weak solution $(\boldsymbol{v}, p)$ of the system (*) completed by Navier's slip boundary conditions such that

$$\boldsymbol{v} \in C(0,T; L^2_{weak}) \cap L^r(0,T; W^{1,r}_{\boldsymbol{n},div})$$

$$p \in L^{\frac{5r}{6}}(0,T; L^{\frac{5r}{6}}) \qquad \text{and} \int_\Omega p(t,x)dx = g(t)$$

If $r \in (9/5, 2)$, the existence of suitable weak solution can be established.

Assumptions on $\boxed{\boldsymbol{S} = \nu(p, e, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})}$ and $\boxed{\boldsymbol{q} = \kappa(p, e, |\boldsymbol{D}(\boldsymbol{v})|^2)\nabla e}$

(B1) given $\boxed{r \in (1, 2)}$ there are $C_1 > 0$ and $C_2 > 0$ and a nonincreasing function $\gamma_1 \in C(\mathbb{R})$, $\gamma_1 \geq 1$, such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$ and all $p \in \mathbb{R}$ and $e \in \mathbb{R}^+$

$$C_1\gamma_1(e)(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \leq \frac{\partial\left[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial\boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2\gamma_1(e)(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2$$

(B2) there is a $\gamma_0 \geq 0$ and a function γ_2 such that for all symmetric matrices $\boldsymbol{D}$ and all p, e

$$\left|\frac{\partial[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial p}\right| \leq \gamma_0\gamma_2(e)(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{4}}$$

(B3) given $\beta > -1$ there are C_4, $C_5 \in (0, \infty)$ such that for all $\boldsymbol{D}$ and all p, e

$$C_4e^{\beta} \leq \kappa(p, e, |\boldsymbol{D}|^2) \leq C_5e^{\beta}$$

Result #3

Theorem 3. (**M. Bulíček**, J. Málek, **K. R. Rajagopal** '07)

Let **(B1)–(B3)** hold and r and β fulfil

$$\boxed{r \in \left(\frac{9}{5}, 2\right) \text{ and } \beta > -\frac{3r-5}{3(r-1)}}$$

with γ_0 "small". Assume that

- $\partial\Omega \in C^{1,1}$
- $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},div}$ and $e_0 \in L^1$, $e_0 \geq C^* > 0$ a.a. in Ω
- $g \in L^{r'}(0,T)$ $\qquad\qquad \int_{\Omega_0} p(t,x)\,dx = g(t)$

Then for all $T > 0$ (and any $\alpha \in (0,1]$) and any $(\boldsymbol{v}_0, e_0)$ there exists at least one suitable weak solution $(\boldsymbol{v}, p, e)$ of the system relevant system completed by Navier's slip boundary conditions (mechanically and thermally isolated domain).

Assumptions on $\boxed{\boldsymbol{S} = \nu(e, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})}$ and $\boxed{\boldsymbol{q} = \kappa(e, \nabla e)\nabla e}$

(C1) given $\boxed{r > 1}$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$ and $e \in \mathbb{R}^+$

$$C_1(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \leq \frac{\partial \left[\nu(e, |\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2(1 + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2$$

(C2) given $\boxed{q > 1}$ there are $C_3 > 0$ and $C_4 > 0$ such that for all vectors $\boldsymbol{u}$, $\boldsymbol{w}$ and $e \in \mathbb{R}^+$

$$C_3(1 + |\boldsymbol{u}|^2)^{\frac{q-2}{2}}|\boldsymbol{w}|^2 \leq \frac{\partial \left[\kappa(e, \boldsymbol{u})\boldsymbol{u}\right]}{\partial \boldsymbol{u}} \cdot (\boldsymbol{w} \otimes \boldsymbol{w}) \leq C_4(1 + |\boldsymbol{u}|^2)^{\frac{q-2}{2}}|\boldsymbol{w}|^2$$

Result #4

Theorem 4. (**M. Bulíček, L. Consiglieri**, J. Málek '07)

Let **(C1)–(C2)** hold and r and q fulfil

$$\boxed{r > \frac{9}{5} \text{ and } q > \frac{7}{4}}$$

Assume that

- $\partial\Omega \in C^{1,1}$
- $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},div}$ and $e_0 \in L^1$, $e_0 \geq C^* > 0$ a.a. in Ω

Then for all $T > 0$ (and any $\alpha \in (0,1]$) and any $(\boldsymbol{v}_0, e_0)$ there exists at least one suitable weak solution $(\boldsymbol{v}, p, e)$ of the system relevant system completed by Navier's slip boundary conditions (mechanically and thermally isolated domain).