Characteristic IBVP's of symmetric hyperbolic systems

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EVEQ 2008, International Summer School on Evolution Equations,
Prague, Czech Republic, 16 - 20. 6. 2008

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Plan

Characteristic hyperbolic IBVP

Consider the problem

$$\begin{cases} Lu = F & \text{in } Q_T\,, \\ Mu = G & \text{on } \Sigma_T\,, \\ u_{|t=0} = f & \text{in } \Omega\,, \end{cases}$$

where

- $\Omega \subset \mathbb{R}^n, Q_T = \Omega \times (0,T), \Sigma_T = \partial\Omega \times (0,T)$
- $L := A_0(x,t,u)\partial_t + \sum_{j=1}^n A_j(x,t,u)\partial_{x_j} + B(x,t,u)$, $A_j, B \in \mathbf{M}_{N\times N}$
- $M = M(x,t) \in \mathbf{M}_{d\times N}$, $\text{rank}(M) = d$ (maximal rank)
- $u(x,t) \in \mathbb{R}^N,\ F(x,t) \in \mathbb{R}^N,\ f(x) \in \mathbb{R}^N,\ G(x,t) \in \mathbb{R}^d$

Characteristic boundary

The boundary $\partial\Omega$ is characteristic if the boundary matrix

$$A_\nu := \sum_{j=1}^{n} A_j \nu_j$$

is singular ar $\partial\Omega$ (not invertible). ($\nu = \nu(x)$ outward normal vector to $\partial\Omega$).
Full regularity (existence in usual Sobolev spaces $H^m(\Omega)$) can't be expected, in general, because of the possible loss of normal regularity at $\partial\Omega$.
[Tsuji, Proc. Japan Acad. 1972],
MHD [Ohno & Shirota, ARMA 1998].

Generally speaking, one normal derivative (w.r.t. $\partial\Omega$) is controlled by two tangential derivatives. Natural function space is the

weighted anisotropic Sobolev space

$$H^m_*(\Omega) := \{u \in L^2(\Omega) : \ Z^\alpha \partial^k_{x_1} u \in L^2(\Omega), \ |\alpha| + 2k \leq m\},$$

where

$$Z_1 = x_1 \partial_{x_1} \quad \text{and} \quad Z_j = \partial_{x_j} \quad \text{for } j = 2, \dots, n,$$

if $\Omega = \{x_1 > 0\}$.

[Chen Shuxing, Chinese Ann. Math. 1982],
[Yanagisawa & Matsumura, CMP 1991].

back to H^m_{tan}
back to $m = 1$

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Known results

Known results have been proved for
Symmetric hyperbolic systems
($A_0, A_1, \ldots, A_n$ are symmetric matrices, A_0 is positive definite),

Maximal non-negative boundary conditions:
($(A_\nu u, u) \geq 0$ for all $u \in ker\, M$, and $ker\, M$ is maximal w.r.t. this property).

- Linear L^2 theory [Rauch, Trans. AMS 1985],
- Existence theory in $H^m_*(\Omega)$ [Guès, CPDE '90], [Ohno, Shizuta, Yanagisawa, JM Kyoto U '95], [Secchi, DIE '95, ARMA '96, Arch. Math. 2000], [Shizuta, Proc. Japan Acad. MS 2000], [Casella, Secchi, Trebeschi, IJPAM 2005, DIE 2006],
- Application to MHD [Secchi, Arch. Math. 1995, NoDEA 2002].

Other known results

Other results for:

Symmetrizable hyperbolic systems under some structural assumptions
▸ St

Uniformly characteristic boundary (the boundary matrix A_ν has constant rank in a neighborhood of $\partial\Omega$)

Uniform Kreiss-Lopatinskii conditions (UKL) ▸ UKL

General theory: [Majda & Osher, CPAM 1975], [Ohkubo, Hokkaido MJ 1981], [Benzoni & Serre, Oxford SP 2007].
Existence of rarefaction waves [Alinhac, CPDE 1989].
Existence of sound waves [Métivier, JMPA 1991].
Elasticity [Morando & Serre, CMS 2005].

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Plan

Compressible vortex sheets

Characteristic free boundary value problems for piecewise smooth solutions: 2D vortex sheets for compressible Euler equations:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \mathbf{u}) = 0\,, \\ \partial_t (\rho\, \mathbf{u}) + \nabla_x \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla_x\, p(\rho) = 0\,, \end{cases} \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^2$.
At the unknown discontinuity front $\Sigma = \{x_1 = \varphi(x_2, t)\}$

$$\partial_t \varphi = v^{\pm} \cdot \nu, \quad [p] = 0,$$

where $[p] = p^+ - p^-$ denotes the jump across it.
Here the mass flux $j = j^{\pm} := \rho^{\pm}(v^{\pm} \cdot \nu - \partial_t \varphi) = 0$ at Σ.

[Coulombel & Secchi, Indiana UMJ 2004, Ann. Sci. ENS 2008].

Strong discontinuities for ideal MHD

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho\, v) = 0\,, \\
\partial_t (\rho\, v) + \nabla \cdot (\rho\, v \otimes v - H \otimes H) + \nabla (p + \frac{1}{2}|H|^2) = 0\,, \\
\partial_t H - \nabla \times (v \times H) = 0\,, \\
\partial_t \big(\rho e + \frac{1}{2}(\rho |v|^2 + |H|^2)\big) \\
\qquad\qquad + \nabla \cdot \big(\rho v (e + \frac{1}{2}|v|^2) + vp + H \times (v \times H)\big) = 0\,, \\
\nabla \cdot H = 0\,,
\end{cases}
$$

ρ density, S entropy,
v velocity field, H magnetic field,
$p = p(\rho, S)$ pressure (such that $p'_\rho > 0$),
$e = e(\rho, S)$ internal energy.

"Gibbs relation"

$$T\,dS = de + p\,dV$$

(T absolute temperature, $V = \dfrac{1}{\rho}$ specific volume) yields

$$p = -\left(\frac{\partial e}{\partial V}\right)_S = \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_S ,$$

$$T = \left(\frac{\partial e}{\partial S}\right)_\rho .$$

We have a closed system for the vector of unknowns (ρ, v, H, S).

Rankine-Hugoniot conditions for MHD

The Rankine-Hugoniot jump conditions at $\Sigma = \{x_1 = \varphi(x_2, x_3, t)\}$ read

$$[j] = 0, \quad [H_N] = 0, \quad j[v_N] + [q]|N|^2 = 0,$$

$$j[v_\tau] = H_N^+[H_\tau], \quad j[H_\tau/\rho] = H_N^+[v_\tau]$$

$$j[e + \frac{1}{2}|v|^2 + \frac{|H|^2}{2\rho}] + [qv_N - H_N(v \cdot H)] = 0,$$

where

$$N = (1, -\partial_{x_2}\varphi, -\partial_{x_3}\varphi) \quad \text{(normal vector)},$$
$$v_N = v \cdot N, \qquad H_N = H \cdot N,$$
$$v_\tau = v - v_N N, \qquad H_\tau = H - H_N N,$$
$$j := \rho(v_N - \partial_t\varphi) \quad \text{(mass flux)},$$
$$q := p + \frac{1}{2}|H|^2 \quad \text{(total pressure)}.$$

Classification of strong discontinuities in MHD:

- MHD shocks:

$$j^{\pm} \neq 0, \quad [\rho] \neq 0,$$

- Alfvén or rotational discontinuities (Alfvén shocks):

$$j^{\pm} \neq 0, \quad [\rho] = 0,$$

- contact discontinuities:

$$j^{\pm} = 0, \quad H_N^{\pm} \neq 0,$$

- current-vortex sheets (tangential discontinuities):

$$j^{\pm} = 0, \quad H_N^{\pm} = 0,$$

Except for MHD shocks, all the above free boundaries are characteristic.

The Rankine-Hugoniot conditions are satisfied as follows:

- Alfvén or rotational discontinuities (Alfvén shocks) $(j^{\pm} \neq 0\,, [\rho] = 0)$:

$$[p] = 0, \quad [S] = 0, \quad [H_N] = 0, \quad [|H|^2] = 0, \quad [v - \frac{H}{\sqrt{\rho}}] = 0,$$

$$j = j^{\pm} = \rho^{\pm}(v_N^{\pm} - \partial_t \varphi) = H_N^{+}\sqrt{\rho^{+}} \neq 0.$$

- Planar Alfvén discontinuities are never uniformly stable (uniform Lopatinskii condition is always violated). They are either weakly stable or violently unstable (Hadamard ill-posedness).
Incompressible MHD [Syrovatskii, 1957], Compressible MHD [Ilin & Trakhinin, Preprint 2007].
- The symbol associated to the front is not elliptic.
- The front is characteristic.

- Contact discontinuities ($j^{\pm} = 0\,,\ H^{\pm}_N \neq 0$):

$$[v] = 0, \quad [H] = 0, \quad [p] = 0.$$

(We may have $[\rho] \neq 0$, $[S] \neq 0$.)

- Boundary conditions are maximally non-negative (but non strictly dissipative).
A priori estimate by the energy method [Blokhin & Trakhinin, Handbook Math. Fluid Dyn. 2002].
- The symbol associated to the front is not elliptic.
- The front is characteristic.

- Current-vortex sheets (tangential discontinuities) ($j^{\pm} = 0,\ H_N^{\pm} = 0$):

$$\partial_t \varphi = v_N^{\pm}, \quad [q] = 0, \quad H_N^{\pm} = 0. \qquad (2)$$

(We may have $[v_\tau] \neq 0,\ [H_\tau] \neq 0,\ [\rho] \neq 0,\ [S] \neq 0.$)

- Planar current-vortex sheets are never uniformly stable (uniform Lopatinskii condition is always violated). They are either weakly stable or violently unstable (Hadamard ill-posedness).
- The symbol associated to the front is elliptic.
- The front is characteristic.

Stability of current-vortex sheets [Trakhinin, ARMA 2005]:

- New symmetrization of the MHD equations,
- Under the assumption $H^+ \times H^- \neq 0$, and a smallness condition on $[v_\tau] \neq 0$, the b.c.s (2) are maximally non-negative (but not strictly dissipative),
- For non-planar current-vortex sheets, prove an a priori estimate by the energy method, without loss of regularity w.r.t. the initial data (but not to the coefficients).

Existence of current-vortex sheets [Trakhinin, ARMA 2008]:

- Tame estimate in anisotropic Sobolev spaces $H^m_*(\Omega)$,
- Nash-Moser iteration.

The above problems are (non standard) characteristic free boundary value problems for symmetrizable hyperbolic systems.

- The boundary conditions may be not maximally non-negative.

- For these problems the Uniform Kreiss-Lopatinskii condition (UKL) is never satisfied. The Kreiss-Lopatinskii condition is either violated (Hadamard ill-posedness) or satisfied in weak form.

Problem of regularity

For general boundary conditions, the current theory is mainly devoted to establish sufficient conditions for the L^2 well-posedness.

We consider the problem of regularity:

Prove the regularity of *any* given L^2 solution, satisfying an apriori energy estimate, for sufficiently smooth data.

(Independently of the structural assumptions on L and M providing the L^2 well-posedness).

Characteristic IBVP

Consider the problem

$$\begin{cases} Lu = F & \text{in } Q_T\,, \\ Mu = G & \text{on } \Sigma_T\,, \\ u_{|t=0} = f & \text{in } \Omega\,, \end{cases} \tag{3}$$

where

- $\Omega \subset \mathbb{R}^n, Q_T = \Omega \times (0,T), \Sigma_T = \partial\Omega \times (0,T)$
- $L := A_0(x,t)\partial_t + \sum_{j=1}^n A_j(x,t)\partial_{x_j} + B(x,t),\ \ A_j, B \in \mathbf{M}_{N\times N}$
- $M = M(x,t) \in \mathbf{M}_{d\times N}$, rank$(M) = d$ (maximal rank)
- $u(x,t) \in \mathbb{R}^N,\ F(x,t) \in \mathbb{R}^N,\ f(x) \in \mathbb{R}^N,\ G(x,t) \in \mathbb{R}^d$

Assumptions

Assume that:

- L is symmetric hyperbolic.
- Characteristic boundary of constant multiplicity: the boundary matrix A_ν has constant rank r at $\partial\Omega$, $0 < r < N$.
- $M(x,t) \in \mathcal{C}^\infty$ and $rank(M) = d$ equals the number of negative eigenvalues of A_ν.
- Reflexivity:

$$ker\, A_\nu \subset ker\, M.$$

- Let $P(x,t)$ be the orthogonal projection onto $ker\, A_\nu(x,t)^\perp$. Then $P(x,t) \in \mathcal{C}^\infty$.

▸ P

- Existence of the L^2-weak solution:

  Assume $A_i \in Lip(\overline{Q}_T)$, for $i = 0, \dots, n$. For all $B \in L^\infty(\overline{Q}_T)$, there exists $\gamma_0 \geq 1$ such that for all $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $f \in L^2(\Omega)$ problem (3) admits a unique solution $u \in C([0,T]; L^2(\Omega))$ with $Pu_{|\,\Sigma_T} \in L^2(\Sigma_T)$.
- u enjoys the energy estimate for all $\gamma \geq \gamma_0$, $0 < \tau \leq T$,

$$e^{-2\gamma\tau}||u(\tau)||^2_{L^2} + \int_0^\tau e^{-2\gamma t}\left(\gamma||u(t)||^2_{L^2} + ||Pu_{|\,\Sigma_T}(t)||^2_{L^2(\partial\Omega)}\right) dt$$

$$\leq C_0\left(||f||^2_{L^2} + \int_0^\tau e^{-2\gamma t}\left(\frac{1}{\gamma}||F(t)||^2_{L^2} + ||G(t)||^2_{L^2(\partial\Omega)}\right) dt\right).$$

Cfr. [Rauch, CPAM 1972].

back

Let us introduce the spaces

$$\mathcal{C}_T(H^m_*) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_*(\Omega)),$$

$$\mathcal{L}^\infty_T(H^m_*) := \bigcap_{j=0}^{m} W^{j,\infty}(0,T; H^{m-j}_*(\Omega)).$$

We denote by $f^{(h)}$ the h^{th} time derivative calculated from (3) at $t=0$ (in terms of $f, F(0), \partial_t F(0), \dots$), and $f^{(0)} = f$. Define

$$|||f|||^2_{m,*} = \sum_{h=0}^{m} \|f^{(h)}\|^2_{H^{m-h}_*(\Omega)}.$$

The compatibility conditions of order $m-1$ are:

$$\sum_{h=0}^{p} \binom{p}{h} (\partial_t^{p-h} M)_{|\,t=0} f^{(h)} = \partial_t^p G_{|\,t=0}\,, \quad \text{on } \partial\Omega,\ p = 0, \dots, m-1\,.$$

Theorem (Morando, S., Trebeschi, 2008)

Let $m \in \mathbb{N}$ *and* $s = \max\{m, [(n+1)/2] + 5\}$.
Assume $A_j \in \mathcal{L}^\infty_T(H^s_*)$, *for* $j = 0, \ldots, n$, *and* $B \in \mathcal{L}^\infty_T(H^s_*)$.
For all $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, $f \in H^m_*(\Omega)$, *with* $f^{(h)} \in H^{m-h}_*(\Omega)$ *for* $h = 1, \ldots, m$, *satisfying the compatibility conditions of order* $m-1$, *the unique solution* u *to* (3) *belongs to* $\mathcal{C}_T(H^m_*)$ *and* $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$.
Moreover u *enjoys the a priori estimate*

$$||u||_{\mathcal{C}_T(H^m_*)} + ||Pu_{|\Sigma_T}||_{H^m(\Sigma_T)} \leq C_m \left(|||f|||_{m,*} + ||F||_{H^m_*(Q_T)} + ||G||_{H^m(\Sigma_T)} \right). \tag{4}$$

Cfr. [Tartakoff, Indiana UMJ 1972]

Plan

Tangential regularity

Reduce locally to the case $(t = x_{n+1})$

- $Q = \mathbb{R}^{n+1}_+ = \{x_1 > 0\}, \;\; \Sigma = \{x_1 = 0\} \times \mathbb{R}^n_{x',t},$
- $\mathrm{supp}(u) \subset \{|x| < 1, \, x_1 \geq 0\}.$

Consider the BVP

$$\begin{cases} (\gamma + L)u_\gamma = F_\gamma & \text{in } \; Q, \\ Mu_\gamma = G_\gamma & \text{on } \Sigma, \end{cases} \tag{5}$$

where $u_\gamma = e^{-\gamma t}u$, $F_\gamma = e^{-\gamma t}F$, $G_\gamma = e^{-\gamma t}G$.
Define the "conormal" Sobolev space

$$H^m_{tan}(Q) = H^m(Q;\Sigma) := \{u \in L^2(Q) : \; Z^\alpha u \in L^2(Q), \; |\alpha| \leq m\}.$$

Go to H^m_*

Theorem (Morando, S., Trebeschi, 2008)

Under all the previous assumptions, there exists $\gamma_m \geq \gamma_0$ *such that, if* $\gamma > \gamma_m$, *and* $F_\gamma \in H^m_{tan}(Q)$, $G_\gamma \in H^m(\Sigma)$, *then* $u_\gamma \in H^m_{tan}(Q)$ *as well, with* $Pu_{\gamma|\Sigma} \in H^m(\Sigma)$. *Moreover* u_γ *enjoys the estimate*

$$\gamma||u_\gamma||^2_{H^m_{tan}(Q)} + ||Pu_{\gamma|\Sigma}||^2_{H^m(\Sigma)} \\ \leq C_m\left(\frac{1}{\gamma}||F_\gamma||^2_{H^m_{tan}(Q)} + ||G_\gamma||^2_{H^m(\Sigma)}\right)$$

where C_m *is independent of* γ, u, F, G.

Cfr. [Rauch, Trans. AMS 1985].

Here the matrices A_j need not to be symmetric.

▸ L^2 estimate

Scheme of the proof

Introduce the (norm preserving) bijection

$$\sharp : L^2(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^{n+1})$$

by

$$w^\sharp(x) := w(e^{x_1}, x')e^{x_1/2}.$$

The map

$$\sharp : H^q_{tan}(\mathbb{R}^{n+1}_+) \to H^q(\mathbb{R}^{n+1})$$

is an isomorphism.

Consider the family of norms

$$|||w|||^2_{\mathbb{R}^{n+1}_+,q-1,tan,\delta} := \|w^\sharp\|^2_{\mathbb{R}^{n+1},q-1,\delta} := \int_{\mathbb{R}^{n+1}} |(w^\sharp)^\wedge(\xi)|^2 \langle\xi\rangle^{2q} \langle\delta\xi\rangle^{-2} d\xi,$$

for $0 < \delta \leq 1$, with $\langle\xi\rangle^2 := 1 + |\xi|^2$. Here $(w^\sharp)^\wedge(\xi)$ denotes the Fourier transform of $w^\sharp(x)$ w.r.t. x.
This norm is equivalent to $\|w\|_{H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)}$ for each fixed $0 < \delta \leq 1$.
Moreover,

$$w \in H^q_{tan}(\mathbb{R}^{n+1}_+)$$

if and only if

$$w \in H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)$$

and

$$|||w|||_{\mathbb{R}^{n+1}_+,q-1,tan,\delta} \quad \text{remains bounded as } \delta \downarrow 0.$$

We define the following mollifier [Nishitani & Takayama, CPDE 2000].

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$. For all $0 < \varepsilon < 1$ set $\chi_\varepsilon(y) := \varepsilon^{-n}\chi(y/\varepsilon)$. We define $J_\varepsilon : L^2(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^{n+1}_+)$ by

$$J_\varepsilon w(x) := \int_{\mathbb{R}^{n+1}} w(x_1 e^{-y_1}, x' - y')e^{-y_1/2}\chi_\varepsilon(y)dy.$$

Then

$$\|J_\varepsilon w\|_{H^q_{tan}(\mathbb{R}^{n+1}_+)} \leq \frac{c}{\varepsilon^q}\|w\|_{L^2(\mathbb{R}^{n+1}_+)} \qquad \forall q \geq 1, \forall \epsilon > 0,$$

$$[Z_j, J_\varepsilon] = 0\,, \quad j = 1, \ldots, n+1.$$

Because of

$$(J_\varepsilon w)^\sharp = w^\sharp * \chi_\varepsilon$$

a result by [Hörmander, 1963] yields

Theorem

Assume that the function $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ *satisfies*

$$\widehat{\chi}(\xi) = O(|\xi|^p) \quad \text{as } \xi \to 0, \quad \text{for some } p \in \mathbb{Z}_+;$$
$$\widehat{\chi}(t\xi) = 0\,, \quad \text{for all } t \in \mathbb{R}\,, \quad \text{implies } \xi = 0.$$

Then for $q \in \mathbb{Z}^+$ *with* $q < p$*, there exists* $C_0 = C_0(\chi, q) > 0$ *such that for all* $0 < \delta \leq 1$ *and* $w \in H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)$

$$\begin{aligned} & C_0^{-1} ||w||^2_{\mathbb{R}^{n+1}_+, q-1, tan, \delta} \\ & \leq \int_0^1 ||J_\varepsilon w||^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon} + ||w||^2_{H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)} \\ & \leq C_0 ||w||^2_{\mathbb{R}^{n+1}_+, q-1, tan, \delta}. \end{aligned}$$

From (5) we infer

$$\begin{cases} (\gamma + L) J_\epsilon u_\gamma = J_\epsilon F_\gamma + [L, J_\epsilon] u_\gamma & \text{in } Q\,, \\ M J_\epsilon u_\gamma = G_\gamma * \widetilde{\chi}_\varepsilon & \text{on } \Sigma\,, \end{cases}$$

where

$$\widetilde{\chi}_\varepsilon(y') := \int_{\mathbb{R}} e^{-y_1/2} \chi_\varepsilon(y_1, y') dy_1\,, \quad y' \in \mathbb{R}^n\,.$$

By assumption

$$\gamma ||J_\varepsilon u_\gamma||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||P J_\varepsilon u_{\gamma|\,\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)} \leq C_0 \left(\frac{1}{\gamma} ||J_\varepsilon F_\gamma + [L, J_\varepsilon] u_\gamma||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||G_\gamma * \widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)} \right). \tag{6}$$

For semplicity, let us remove the subscript γ from $u_\gamma, F_\gamma, G_\gamma$.

$F \in H^q_{tan}(\mathbb{R}^{n+1}_+),\ q \leq m$, yields

$$\int_0^1 ||J_\varepsilon F||^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon} \leq C||F||^2_{\mathbb{R}^{n+1}_+,q,tan,1},$$

for all $0 < \delta \leq 1$.
Moreover, $G \in H^q(\mathbb{R}^n),\ q \leq m$, yields

$$\int_0^1 ||G * \widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon} \leq C||G||^2_{\mathbb{R}^n,q},$$

for all $0 < \delta \leq 1$.
We need to estimate the commutator in (6).

Lemma

If $u \in L^2(\mathbb{R}^{n+1}_+)$ *and if* $a(x) \in \mathcal{C}^\infty_{(0)}(\mathbb{R}^{n+1}_+)$*, then* $([a, J_\varepsilon]u)^\sharp$ *can be written as*

$$\int_{\mathbb{R}^{n+1}} b(x,y) u^\sharp(x-y) y^\alpha \chi_\varepsilon(y) dy, \qquad |\alpha| = 1.$$

For $j = 1, \cdots, n+1$, $([aZ_j, J_\varepsilon]u)^\sharp$ *can be written as sum of terms of the form*

$$\int_{\mathbb{R}^{n+1}} b(x,y) u^\sharp(x-y) \chi_\varepsilon(y) dy,$$

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^{n+1}} b(x,y) u^\sharp(x-y) y^\alpha (\partial_{x_j} \chi)_\varepsilon(y) dy, \qquad |\alpha| = 1.$$

Here $b(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$*, the space of* $\mathcal{C}^\infty$ *functions with bounded derivatives.*

From the previous lemma one has

Lemma (Nishitani & Takayama, 2000)

Let $a \in \mathcal{C}^\infty_{(0)}(\mathbb{R}^{n+1}_+)$ *and* $q \geq 1$. *Then there exists a constant* $C > 0$ *such that for all* $0 < \delta \leq 1$

$$\int_0^1 ||[a, J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon} \leq C||u||^2_{\mathbb{R}^{n+1}_+, q-2, tan, \delta} ,$$

and, for $j = 1, \cdots, n+1$,

$$\int_0^1 ||[aZ_j, J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon} \leq C||u||^2_{\mathbb{R}^{n+1}_+, q-1, tan, \delta} .$$

(This suffices for tangential derivatives)

The commutator $[A_1\partial_1, J_\varepsilon]$

Reduce locally to the case

$$-A_\nu = A_1 = \begin{pmatrix} A_1^{I\,I} & A_1^{I\,II} \\ A_1^{II\,I} & A_1^{II\,II} \end{pmatrix},$$

where

back to A_1

$$A_1^{I\,I} \in \mathbf{M}_{r\times r} \qquad \text{is invertible},$$

and

$$A_1^{I\,II} = 0, \quad A_1^{II\,I} = 0, \quad A_1^{II\,II} = 0 \qquad \text{at } \Sigma = \{x_1 = 0\}.$$

Decompose accordingly $u = (u^I, u^{II})$. Then $Pu = u^I$.

From (3) we infer ($\partial_{x_{n+1}} = \partial_t,\ A_{n+1} = I$)

$$\partial_{x_1} u^I = -(A_1^{I\,I})^{-1} \left(A_1^{I\,II} \partial_{x_1} u^{II} + \Big(\gamma u + \sum_{j=2}^{n+1} A_j \partial_{x_j} u + Bu - F \Big)^I \right)$$

where $A_1^{I\,II} \partial_{x_1} u^{II}$ behaves like $Z_1 u$. Therefore $\partial_{x_1} u^I$ is controlled by only tangential derivatives.
The other normal derivatives in L are

$$A_1^{II\,I} \partial_{x_1} u^I, \quad A_1^{II\,II} \partial_{x_1} u^{II}$$

which also behave like $Z_1 u$.

We obtain

Lemma

Let $q = 1, \ldots, m$. There exists a constant $C > 0$ such that

$$\int_0^1 \;||[A_1\partial_1, J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2q}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$$

$$\leq C\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2q}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$$

$$+C\gamma^2||u||^2_{H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)} + C||F||^2_{\mathbb{R}^{n+1}_+,q-2,tan,\delta}.$$

for all $0 < \delta \leq 1$ and for γ large enough.

Combining all the previous estimates gives for $\gamma > \gamma_q$, where $\gamma_q \geq \gamma_0$ is large enough,

$$\gamma \int_0^1 ||J_\varepsilon u_\gamma||^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}$$
$$+ \int_0^1 ||J_\varepsilon u^I_{\gamma|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2q} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}$$
$$\leq C \left(\frac{1}{\gamma} ||F_\gamma||^2_{\mathbb{R}^{n+1}_+,q,tan} + ||G_\gamma||^2_{\mathbb{R}^n,q} + \gamma ||u_\gamma||^2_{\mathbb{R}^{n+1}_+,q-1,tan} \right),$$

for all $0 < \delta \leq 1$ and $q = 1, \ldots, m$.
Therefore, if $u_\gamma \in H^{q-1}_{tan}(\mathbb{R}^{n+1}_+)$ we infer that

$$||u_\gamma||^2_{\mathbb{R}^{n+1}_+,q-1,tan,\delta} + ||u^I_{\gamma|\{x_1=0\}}||^2_{\mathbb{R}^n,q-1,\delta}$$

is uniformly bounded in δ.
Then

$$u_\gamma \in H^q_{tan}(\mathbb{R}^{n+1}_+), \quad u^I_{\gamma|\{x_1=0\}} \in H^q(\mathbb{R}^n).$$

By induction we thus obtain $u_\gamma \in H^m_{tan}(\mathbb{R}^{n+1}_+)$ with $u^I_{\gamma|\{x_1=0\}} \in H^m(\mathbb{R}^n)$

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The homogeneous IBVP

From the result on tangential regularity we infer for the solution u_γ of the homogeneous IBVP

$$\begin{cases} L_\gamma u_\gamma = F_\gamma & \text{in } Q_T\,, \\ M u_\gamma = G_\gamma & \text{on } \Sigma_T\,, \\ u_{\gamma|t=0} = 0 & \text{in } \Omega\,, \end{cases} \tag{7}$$

that

$$u_\gamma \in H^m_{tan}(Q_{T'}) \quad \text{with} \quad P u_{\gamma|\Sigma_T} \in H^m(\Sigma_{T'})$$

($\forall\, T' < T$), provided that

$$\partial_t^h F_{\gamma|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma|\,t=0} = 0\,, \quad h = 0, \dots, m-1\,.$$

The nonhomogeneous IBVP for $m = 1$

Consider the nonhomogeneous IBVP

$$\begin{aligned} L_\gamma u_\gamma &= F_\gamma && \text{in } Q_T\,, \\ M u_\gamma &= G_\gamma && \text{on } \Sigma_T\,, \\ u_{\gamma|t=0} &= f && \text{in } \Omega\,. \end{aligned} \tag{8}$$

Now we look for an approximated solution u_k of (8) of the form $u_k = v_k + w_k$, where v_k is solution to

$$\begin{aligned} Lv_k &= F_k - Lw_k, && \text{in } Q_T \\ Mv_k &= G_k - Mw_k, && \text{on } \Sigma_T \\ v_{k|t=0} &= 0, && \text{in } \Omega. \end{aligned} \tag{9}$$

Let us denote $u_{k\gamma} = e^{-\gamma t} u_k$, $v_{k\gamma} = e^{-\gamma t} v_k$ and so on. Then (9) is equivalent to

$$\begin{array}{ll} L_\gamma v_{k\gamma} = F_{k\gamma} - L_\gamma w_{k\gamma}, & \text{in } Q_T \\ M v_{k\gamma} = G_{k\gamma} - M w_{k\gamma}, & \text{on } \Sigma_T \\ v_{k\gamma|t=0} = 0, & \text{in } \Omega. \end{array} \tag{10}$$

We look for $w_{k\gamma}$ such that

$$\begin{aligned} (F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} \ &= \ \partial_t (F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} = 0, \\ (G_{k\gamma} - M w_{k\gamma})_{|t=0} &= \ 0 \ \partial_t (G_{k\gamma} - M w_{k\gamma})_{|t=0} = 0. \end{aligned} \tag{11}$$

Regularization of the data

Lemma

Let $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$, $f \in H^1_*(\Omega)$, *with* $f^{(1)} \in L^2(\Omega)$, *such that* $Mf_{|\partial\Omega} = G_{|t=0}$.
Then there exist $F_k \in H^3(Q_T)$, $G_k \in H^3(\Sigma_T)$, $f_k \in H^3(\Omega)$, *such that* $M(0)f_k = G_k(0)$, $\partial_t M(0)f_k + M(0)f_k^{(1)} = \partial_t G_k(0)$ *on* $\partial\Omega$, *and such that* $F_k \to F$ *in* $H^1_*(Q_T)$, $G_k \to G$ *in* $H^1(\Sigma_T)$, $f_k \to f$ *in* $H^1_*(\Omega)$, $f_k^{(1)} \to f^{(1)}$ *in* $L^2(\Omega)$, *as* $k \to +\infty$.

It seems that this Lemma can be proved in H^m_* only for $m = 1$.

▸ Go to H^m_*

Then we take a function $w_k \in H^3(Q_T)$ such that

$$w_{k|t=0} = f_k, \quad \partial_t w_{k|t=0} = f_k^{(1)}, \quad \partial^2_{tt} w_{k|t=0} = f_k^{(2)}.$$

Notice that this yields

$$(Lw_k)_{|t=0} = F_{k|t=0}, \quad \partial_t(Lw_k)_{|t=0} = \partial_t F_{k|t=0},$$

i.e.

$$(F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} = 0, \quad \partial_t(F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} = 0,$$

and

$$M(0)f_{k|\partial\Omega} = G_{k|t=0}, \quad \partial_t M(0)f_{k|\partial\Omega} + M(0)f^{(1)}_{k|\partial\Omega} = \partial_t G_{k|t=0},$$

yields

$$(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0, \quad \partial_t(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0.$$

Thus we have (11) and we may deduce $u_k \in H^2_{tan}(Q_{T'})$, with $Pu_{k|\Sigma_{T'}} \in H^2(\Sigma_{T'})$, $\forall\, T' < T$.

A priori estimate in H^1_*

In local coordinates one shows that the commutator $[L, Z_i]$ contains only tangential derivatives:

there exist matrices $\Gamma_\beta, \Gamma_0, \Psi$ such that

$$[L, Z_i] = -\sum_{|\beta|=1} \Gamma_\beta Z^\beta + \Gamma_0 + \Psi L, \qquad i = 1, \dots, n+1.$$

Then Zu_k solves the problem

$$\begin{aligned} &LZ_iu_k + \sum_{|\beta|=1} \Gamma_\beta Z^\beta u_k = (Z_i + \Psi)F_k + \Gamma_0 u_k, && \text{in } \mathbb{R}^n_+ \times]0, T'[, \\ &MZ_iu_k = Z_iG_k, && \text{on } \{x_1 = 0\}, \\ &Z_iu_{k|t=0} = Z_if_k, && \text{in } \mathbb{R}^n_+. \end{aligned}$$

We may apply the L^2 estimate, find an a priori estimate in $\mathcal{C}_T(H^1_*)$, extend up to T, pass to the limit, ... This concludes the proof for $m = 1$.

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Normal regularity for $m \geq 2$

By induction, assume that $u \in \mathcal{C}_T(H_*^{m-1})$.
We need to increase the regularity by one more tangential derivative and, if m is even, also by one more normal derivative.

We consider:

- Only purely tangential derivatives
- Tangential derivatives and only one normal derivative
- Tangential derivatives and more than one normal derivative

Recall that in the space $H_*^m(\Omega)$ tangential derivatives have "weight one" and normal derivatives have "weight two".

Purely tangential derivatives

In local coordinates we decompose A_1 (Go to A_1) and accordingly $\partial_1 u = \begin{pmatrix} \partial_1 u^I \\ \partial_1 u^{II} \end{pmatrix}$.

By inverting $A_1^{I,I}$, we can write $\partial_1 u^I$ as

$$\partial_1 u^I = \Lambda Z u + R \tag{12}$$

where

$$\Lambda Z u = (A_1^{I,I})^{-1} \big[(A_{n+1} Z_{n+1} u + \sum_{j=2}^{n} A_j Z_j u)^I + A_1^{I,II} \partial_1 u^{II} \big],$$
$$R = (A_1^{I,I})^{-1} (Bu - F)^I.$$

Since $A_1^{I\,II} = 0$ at $\{x_1 = 0\}$, $A_1^{I\,II} \partial_{x_1} u^{II}$ behaves like $Z_1 u$. Therefore $\partial_{x_1} u^I$ also behaves like a first order tangential derivative.

Applying the operator Z^α, $|\alpha| = m - 1$, to (3) and substituting (12) gives a problem with the form

$$\begin{array}{ll} (\mathcal{L} + \mathcal{B})Z^\alpha u = \mathcal{F}_\alpha & \text{in } \mathbb{R}^n_+ \times]0, T[, \\ \mathcal{M} Z^\alpha u = Z^\alpha G & \text{on } \{x_1 = 0\}, \\ Z^\alpha u|_{t=0} = f_\alpha & \text{in } \mathbb{R}^n_+, \end{array}$$

where

$$\mathcal{L} = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}.$$

We may apply Theorem 1 for $m = 1$ and infer $Z^\alpha u \in \mathcal{C}_T(H^1_*)$, for all $|\alpha| = m - 1$.

Tangential and one normal derivatives

We apply to the part II of $(3)_1$ the operator $Z^\gamma \partial_1$, with $|\gamma| = m - 2$. We obtain

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}}) Z^\gamma \partial_1 u^{II} = \mathcal{G},$$

where

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{L} & & \\ & \ddots & \\ & & \tilde{L} \end{pmatrix}$$

with $\tilde{L} = A_0^{II,II} \partial_t + \sum_{j=1}^n A_j^{II,II} \partial_j$.

- The boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1 = 0\}$. We don't need any boundary condition.
- $\mathcal{G}$ contains only tangential derivatives of order at most m.

We deduce $Z^\gamma \partial_1 u \in C_T(L^2)$, for all $|\gamma| = m - 2$.

More than one normal derivative

Again by induction.
Suppose that for $1 \leq k < [m/2]$, it has already been shown that $Z^\alpha \partial_1^h u \in C_T(L^2)$, for every $h = 1, \cdots, k$, $|\alpha| + 2h \leq m$.
From (12) we infer

$$Z^\alpha \partial_1^{k+1} u^I \in C_T(L^2).$$

It rests to prove that $Z^\alpha \partial_1^{k+1} u^{II} \in C_T(L^2)$.
We apply operator $Z^\alpha \partial_1^{k+1}, |\alpha| + 2k = m - 2$ to the part II of $(3)_1$ and obtain

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}}_k) Z^\alpha \partial_1^{k+1} u^{II} = \mathcal{G}_k.$$

- $\mathcal{G}_k$ contains derivatives of u of order m, but normal derivatives of order at most k.
- The boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1 = 0\}$.

We infer that

$$Z^\alpha \partial_1^{k+1} u^{II} \in C_T(L^2)$$

for all α, k with $|\alpha| + 2k = m - 2$.

By repeating this procedure we obtain the result for any $k \leq [m/2]$, hence $u \in \mathcal{C}_T(H^m_*)$.

The end!

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Euler equations

$$
\begin{cases}
\frac{\rho_p}{\rho}(\partial_t p + v\cdot\nabla p) + \nabla\cdot v = 0,\\
\rho\{\partial_t v + (v\cdot\nabla)v\} + \nabla p = 0,\\
\partial_t S + v\cdot\nabla S = 0.
\end{cases}
$$

This is a quasi-linear symmetric hyperbolic system since it can be written in the form

$$
\begin{pmatrix}
(\rho_p/\rho)(\partial_t + v\cdot\nabla) & \nabla\cdot & 0\\
\nabla & \rho(\partial_t + v\cdot\nabla)I_3 & \underline{0}\\
0 & \underline{0}^T & \partial_t + v\cdot\nabla
\end{pmatrix}
\begin{pmatrix} p\\ v\\ S\end{pmatrix} = 0.
$$

Boundary matrix:

$$A_\nu = \begin{pmatrix} (\rho_p/\rho) v\cdot\nu & \nu^T & 0 \\ \nu & \rho v\cdot\nu I_3 & \underline{0} \\ 0 & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

If $v\cdot\nu = 0$, then

$$ker\, A_\nu = \{U' = (p', v', S') \,:\, p' = 0,\ v'\cdot\nu = 0\},$$

Projection onto $(ker\, A_\nu)^\perp$:

$$P = \begin{pmatrix} 1 & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & \underline{0} \\ 0 & \underline{0}^T & 0 \end{pmatrix}.$$

P has the regularity of ν.

back

Ideal Magneto-hydrodynamics

$$
\begin{cases}
\rho_p(\partial_t + v \cdot \nabla)p + \rho \nabla \cdot v = 0, \\
\rho\{\partial_t v + (v \cdot \nabla)v\} + \nabla p + \mu H \times (\nabla \times H) = 0, \\
\partial_t H + (v \cdot \nabla)H - (H \cdot \nabla)v + H \nabla \cdot v = 0, \\
\partial_t S + v \cdot \nabla S = 0, \\
\nabla \cdot H = 0.
\end{cases}
$$

The constraint $\nabla \cdot H = 0$ may be considered as a restriction on the initial data.

This is a quasi-linear symmetric hyperbolic system:

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ \underline{0} & 0_3 & I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho) v\cdot\nabla & \nabla\cdot & \underline{0}^T & 0 \\ \nabla & \rho v\cdot\nabla I_3 & \nabla(\cdot)\cdot H - H\cdot\nabla I_3 & \underline{0} \\ \underline{0} & H\nabla\cdot - H\cdot\nabla I_3 & v\cdot\nabla I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nabla \end{pmatrix} \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} = 0$$

A different symmetrization with the total pressure $q = p + |H|^2/2$:

$$\begin{cases} \frac{\rho_p}{\rho}\{(\partial_t + v\cdot\nabla)q - H\cdot(\partial_t + (v\cdot\nabla))H\} + \nabla\cdot v = 0, \\ \rho(\partial_t + (v\cdot\nabla))v + \nabla q - (H\cdot\nabla)H = 0, \\ (\partial_t + (v\cdot\nabla))H - (H\cdot\nabla)v - \\ \qquad -\frac{\rho_p}{\rho}H\{(\partial_t + v\cdot\nabla)q - H\cdot(\partial_t + (v\cdot\nabla))H\} = 0, \\ \partial_t S + v\cdot\nabla S = 0, \end{cases}$$

that we rewrite as

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho)H^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ -(\rho_p/\rho)H & 0_3 & a_0 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho)v\cdot\nabla & \nabla\cdot & -(\rho_p/\rho)H^T v\cdot\nabla & 0 \\ \nabla & \rho v\cdot\nabla I_3 & -H\cdot\nabla I_3 & \underline{0} \\ -(\rho_p/\rho)Hv\cdot\nabla & -H\cdot\nabla I_3 & a_0 v\cdot\nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nabla \end{pmatrix} \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} = 0$$

where

$$a_0 = I_3 + (\rho_p/\rho)H \otimes H.$$

Boundary matrix:

$$A_\nu = \begin{pmatrix} (\rho_p/\rho)v\cdot\nu & \nu^T & -(\rho_p/\rho)H^T v\cdot\nu & 0 \\ \nu & \rho v\cdot\nu I_3 & -H\cdot\nu I_3 & \underline{0} \\ -(\rho_p/\rho)Hv\cdot\nu & -H\cdot\nu I_3 & a_0 v\cdot\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

- If $v\cdot\nu=0,\ H\cdot\nu=0$, then

$$ker\ A_\nu = \{U'=(q',v',H',S')\ :\ q'=0,\ v'\cdot\nu=0\},$$

Projection onto $(ker\ A_\nu)^\perp$:

$$P = \begin{pmatrix} 1 & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & 0_3 & \underline{0} \\ \underline{0} & 0_3 & 0_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix}.$$

- If $H \cdot \nu = 0$ and $v \cdot \nu \neq 0,\ v \cdot \nu \neq \frac{|H|}{\sqrt{\rho}} \pm c(\rho)$, then

$$ker\, A_\nu = \{0\}, \quad P = Id.$$

(Non characteristic boundary)

- If $v \cdot \nu = 0$ and $H \cdot \nu \neq 0$, then

$$\begin{aligned} &ker\, A_\nu = \{v' = 0,\ \nu q' - H \cdot \nu H' = 0\}, \\ &rank\, A_\nu = 6. \end{aligned}$$

Projection onto $(ker\, A_\nu)^\perp$:

$$P = \begin{pmatrix} \Lambda & \underline{0}^T & -\Lambda(H\cdot\nu)\nu^T & 0 \\ \underline{0} & I_3 & 0_3 & \underline{0} \\ -\Lambda(H\cdot\nu)\nu & 0_3 & I_3 - \Lambda\nu\otimes\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix}.$$

where $\Lambda := [1 + (H \cdot \nu)^2]^{-1}$.

P has the (finite) regularity of $H \cdot \nu$ (for $\partial\Omega \in C^\infty$). However, ther is full regularity (solvability in H^m) [Yanagisawa, Hokkaido MJ 1987].

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Kreiss-Lopatinskii condition

Consider the BVP

$$\begin{cases} Lu = F\,, & \text{in}\,\{x_1 > 0\}\,,\\ Mu = G\,, & \text{on}\,\{x_1 = 0\}\,. \end{cases} \tag{13}$$

- $L := \partial_t + \sum_{j=1}^n A_j \partial_{x_j}$, hyperbolic operator (with eigenvalues of constant multiplicity);
- $A_j \in \mathbf{M}_{N\times N}$, $j = 1, \dots, n$, and $\det A_1 \neq 0$ (i.e. non characteristic boundary);
- $M \in \mathbf{M}_{d\times N}$, rank$(M) = d = \#\{\text{positive eigenvalues of } A_1\}$.

- Let $u = u(x_1, x', t)$ $(x' = (x_2, \dots, x_n))$ be a solution to (13) for $F = 0$ and $G = 0$.
- Let $\widehat{u} = \widehat{u}(x_1, \eta, \tau)$ be Fourier-Laplace transform of u w.r.t. x' and t respectively (η and τ dual variables of x' and t respectively).
- $\widehat{u}$ solves the ODE problem

$$\begin{cases} \frac{d\widehat{u}}{dx_1} = \mathcal{A}(\eta, \tau)\widehat{u}\,, \quad x_1 > 0\,, \\ M\widehat{u}(0) = 0\,, \end{cases} \tag{14}$$

where $\mathcal{A}(\eta, \tau) := -(A_1)^{-1} \left(\tau I_n + i \sum_{j=2}^{n} A_j \eta_j \right)$.

Let $\mathcal{E}^-(\eta,\tau)$ be the stable subspace of (14).

- Kreiss-Lopatinkii condition (KL):

$$ker M \cap \mathcal{E}^-(\eta,\tau) = \{0\}, \quad \forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0.$$

$$\Updownarrow$$

$$\forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0,\ \exists C = C(\eta,\tau) > 0\ : \\ |A_1 V| \le C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau).$$

- Uniform Kreiss-Lopatinskii condition (UKL):

$$\exists C > 0 :\ \forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0 : \\ |A_1 V| \le C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau).$$

Lopatinskii determinant

• For all $(\eta, \tau) \in \mathbb{R}^{n-1} \times \mathbb{C}$, $\Re\tau > 0$, let $\{X_1(\eta, \tau), \dots, X_d(\eta, \tau)\}$ be an orthonormal basis of $\mathcal{E}^-(\eta, \tau)$ $(\dim \mathcal{E}^-(\eta, \tau) = \text{rank}\, M = d)$.
• Constant multiplicity of the eigenvalues $\Rightarrow X_j(\eta, \tau)$, $j = 1, \dots, d$, then $\mathcal{E}^-(\eta, \tau)$ can be extended to all $(\eta, \tau) \neq (0, 0)$ with $\Re\tau = 0$.

$$\Delta(\eta, \tau) := \det \left[M \left(X_1(\eta, \tau), \dots, X_d(\eta, \tau) \right) \right] \quad \forall(\eta, \tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau \geq 0.$$

$$(KL) \quad \Leftrightarrow \quad \Delta(\eta, \tau) \neq 0\,, \quad \forall \Re\tau > 0, \forall \eta \in \mathbb{R}^{n-1}\,.$$

$$(UKL) \quad \Leftrightarrow \quad \Delta(\eta, \tau) \neq 0\,, \quad \forall \underline{\Re\tau \geq 0}, \forall \eta \in \mathbb{R}^{n-1}\,.$$

Kreiss-Lopatinskii condition and well posedness

1. $\boxed{\det A_1 \neq 0}$ (i.e. non characteristic boundary)
   - (UKL) $\Leftrightarrow$ L^2-strong well posedness of (13);
   - (KL) but NOT (UKL) $\Rightarrow$ Weak well posedness of (13) (energy estimate with loss of regularity);
   - NOT (KL) $\Rightarrow$ (13) is ill posed in Hadamard's sense.
2. $\boxed{\det A_1 = 0}$ (i.e. characteristic boundary)

   (UKL) $+$ structural assumptions on $L \Rightarrow L^2-$strong well posedness of (13).

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Structural assumptions

- [Majda & Osher, 1975]:
  1. L symmetric hyperbolic, with variable coefficients +
  2. Uniformly characteristic boundary +
  3. (UKL) +
  4. Several structural assumptions on L and M, among which that:

$$A(\eta) := \sum_{j=2}^{n} A_j \eta_j = \begin{pmatrix} a_1(\eta) & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}$$

where $a_1(\eta)$ has only simple eigenvalues for $|\eta| = 1$.

Satisfied by: strictly hyperbolic systems, MHD, Maxwell's equations, linearized shallow water equations.
NOT satisfied by: 3D isotropic elasticity ($a_1(\eta) = 0_3$).

- [Benzoni-Gavage & Serre, 2003]:
  1. L symmetric hyperbolic, with constant coefficients, M constant $+$
  2. Characteristic boundary $+$
  3. (UKL) $+$
  4. $$A(\eta) = \begin{pmatrix} 0 & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}$$

     with $a_2(\eta) = 0$.

  Satisfied by: electromagnetism, Maxwell's equations, acoustics.
  NOT satisfied by: isotropic elasticity ($a_2(\eta) \neq 0$).
- [Morando & Serre, 2005]: $2D, 3D$ linear isotropic elasticity.

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