The Boltzmann equation and its formal hydrodynamic limits

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The Boltzmann equation

▶ A kinetic model for perfect gases

- a **statistical description** of the microscopic state of the gas

$$f \equiv f(t,x,v)$$

density of particles having position x and velocity v at time t

- an evolution driven by **binary interactions**

$$\underbrace{\partial_t f + v\cdot\nabla_x f}_{\text{free transport}} = \underbrace{Q(f,f)}_{\text{localized elastic collisions}}$$

$$Q(f,f) = \iint \big[\underbrace{f(v')f(v'_*)}_{\text{gain term}} - \underbrace{f(v)f(v_*)}_{\text{loss term}}\big]\, b(v-v_*,\omega)dv_*d\omega$$

The **pre-collisional velocities** v' and v'_* are parametrized by $\omega \in S^2$

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

The **cross-section** b depends on the microscopic potential of interaction

$$b(v - v_*, \omega) = |(v - v_*) \cdot \omega| \text{ for hard spheres}$$

It is supposed to satisfy **Grad's cutoff assumption**

$$\begin{aligned} &0 < b(z, \omega) \leq C_b (1 + |z|)^\beta |\cos(\widehat{z, \omega})| \text{ a.e. on } \mathbb{R}^3 \times S^2 \,, \\ &\int_{S^2} b(z, \omega) d\omega \geq \frac{1}{C_b} \frac{|z|}{1 + |z|} \text{ a.e. on } \mathbb{R}^3 \,. \end{aligned}$$

▶ The conservation laws

- **Symmetries of the collision operator**

- the pre-post collisional change of variables

$$(v', v'_*, \omega) \mapsto (v, v_*, \omega)$$

is involutive, has unit Jacobian ;

- it leaves the cross-section invariant

$$b(v - v_*, \omega) \equiv b(|v - v_*|, |(v - v_*) \cdot \omega|)$$

Therefore

$$\int Q(f,f)\varphi(v)dv = \frac{1}{4} \iiint b(v-v_*,\omega)(f'f'_* - ff_*)(\varphi + \varphi_* - \varphi' - \varphi'_*)dvdv_*d\omega$$

- The **collision invariants**

$$\forall f \in C_c(\mathbb{R}^3), \int_{\mathbb{R}^3} Q(f,f)\varphi(v)dv = 0 \quad \Leftrightarrow \quad \varphi \in \text{Vect}\{1, v_1, v_2, v_3, |v|^2\}.$$

- **Local conservation of mass, momentum and energy**

$$\partial_t \int_{\mathbb{R}^3} fdv + \nabla_x \cdot \int_{\mathbb{R}^3} vfdv = 0,$$
$$\partial_t \int_{\mathbb{R}^3} vfdv + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes vfdv = 0,$$
$$\partial_t \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 fdv + \nabla_x \cdot \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 vfdv = 0,$$

- reminiscent of the Euler equations for compressible perfect gases

$$\partial_t R + \nabla_x \cdot (RU) = 0,$$

$$\partial_t (RU) + \nabla_x \cdot (RU \otimes U + P) = 0,$$

$$\partial_t \frac{1}{2}(R|U|^2 + \mathrm{Tr}(P)) + \nabla_x \cdot \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 vfdv = 0$$

▶ Boltzmann's H-theorem

- **Symmetries of the collision operator**

$$D(f) \stackrel{\text{def}}{=} -\int Q(f,f)\log f dv$$
$$= \frac{1}{4}\int B(v-v_*,\omega)(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}dvdv_*d\omega \geq 0$$

- **Thermodynamic equilibria**

Let $f \in C(\mathbb{R}^3)$ such that

$$\int fdv = R, \int vfdv = RU \text{ and } \int |v|^2 fdv = R(|U|^2+3T).$$

Then

$$D(f) = 0 \Leftrightarrow f \text{ minimizer of } \int f\log fdv$$
$$\Leftrightarrow f(v) = \mathcal{M}_{R,U,T}(v) = \frac{R}{(2\pi T)^{3/2}}\exp\left(-\frac{|v-U|^2}{2T}\right)$$

- **Local decay of entropy**

$$\partial_t \int_{\mathbb{R}^3} f \log f dv + \nabla_x \cdot \int_{\mathbb{R}^3} vf \log f dv = \int Q(f, f) \log f \leq 0,$$

- reminiscent of Lax-Friedrichs criterion that selects admissible solutions of the compressible Euler equations

$$\partial_t S + U \cdot \nabla_x S \leq 0,$$

$$S = \log \frac{\mathrm{Tr}(P)}{R}.$$

- suggests that $f(t)$ should relax towards global (in x) thermodynamic equilibrium as $t \to \infty$.

Hydrodynamic regimes

▶ Physical parameters and scalings

Length scales

- l_o observation length scale (macroscopic)
- λ mean free path (mesoscopic)
- δl size of the particles (microscopic) neglected

Velocity scales

- u_o bulk velocity (macroscopic)
- c thermal speed (mesoscopic) related to the temperature T

Time scales

- t_o observation time scale (macroscopic)
- τ average time between two collisions (mesoscopic) related to the density ρ
- δt duration of a collision process (microscopic) neglected

Nondimensional parameters

- the Mach number $\mathrm{Ma} = \frac{u_o}{c}$ measures the compressibility of the gas
- the Strouhal number $\mathrm{St} = \frac{l_o}{ct_o}$

$$\mathrm{Ma} = \mathrm{St} \text{ in the sequel (nonlinear dynamics)}$$

- the Knudsen number $\mathrm{Kn} = \frac{\lambda}{l_o}$ measures the adiabaticity of the gas
- the Reynolds number Re measures the viscosity of the gas

$$\mathrm{Re} = \frac{\mathrm{Ma}}{\mathrm{Kn}} \text{ for perfect gases}$$

Nondimensional form of the Boltzmann equation

$$\mathrm{St}\partial_t f + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f, f)$$

▶ Qualitative behaviour of the Boltzmann equation

Fast relaxation asymptotics $\mathrm{Kn} \to 0$

local thermodynamic equilibrium is reached almost instantaneously

$$f(t,x,v) \sim \frac{R(t,x)}{(2\pi T(t,x))^{3/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2T(t,x)}\right)$$

the state of the gas is determined by the thermodynamic fields R, U, T
$\Rightarrow$ the Knudsen number Kn governs the transition from kinetic theory to fluid dynamics

Main features of the macroscopic flow

- $\mathrm{Ma} \sim 1$: compressible inviscid
- $\mathrm{Ma} << 1$ and $\mathrm{Ma} >> \mathrm{Kn}$: incompressible inviscid
- $\mathrm{Ma} \sim \mathrm{Kn}$: incompressible viscous

$\Rightarrow$ the Mach number Ma determines the fluid regime

▶ The compressible Euler limit

Hilbert's (formal) expansion

$$f(t,x,v) = \mathcal{M}_f(t,x,v) + O(\mathrm{Kn})$$

(provided that Q satisfies good relaxation estimates)

The asymptotic conservation laws

$$\begin{aligned}
&\mathrm{Ma}\partial_t R + \nabla_x \cdot (RU) = 0\\
&\mathrm{Ma}\partial_t RU + \nabla_x \cdot (RU \otimes U + RT) = O(\mathrm{Kn})\\
&\mathrm{Ma}\partial_t \left(\frac{1}{2}R|U|^2 + \frac{3}{2}RT\right) + \nabla_x \cdot \left(\frac{1}{2}R|U|^2 U + \frac{5}{2}RTU\right) = O(\mathrm{Kn})
\end{aligned}$$

(computing the moments of $\mathcal{M}_{R,U,T}$, i.e. the pressure and the energy flux, in terms of the thermodynamic fields R, U, T)

▶ Corrections to the first hydrodynamic approximation

Chapman-Enskog's expansion

$$f(t,x,v) = \mathcal{M}_f(t,x,v)\left(1 + \sum_{k\geq 1}(\mathrm{Kn})^k g_k(t,x,v)\right)$$

(requires knowing in advance that the successive corrections are systems of local conservation laws)

The Navier-Stokes equations

$$\begin{aligned}
&\mathrm{Ma}\partial_t R + \nabla_x\cdot(RU) = 0\\
&\mathrm{Ma}\partial_t RU + \nabla_x\cdot(RU\otimes U + RT) = \mathrm{Kn}\nabla_x\cdot(\mu(R,T)DU)\\
&\mathrm{Ma}\partial_t\left(\frac{1}{2}R|U|^2 + \frac{3}{2}RT\right) + \nabla_x\cdot\left(\frac{1}{2}R|U|^2U + \frac{5}{2}RTU\right)\\
&\qquad = \mathrm{Kn}\nabla_x\cdot(\kappa(R,T)\nabla_x T) + \mathrm{Kn}\nabla_x\cdot(\mu(R,T)DU\cdot U)
\end{aligned}$$

obtained by solving the Fredholm equation

$$2Q(\mathcal{M}_f,\mathcal{M}_f g_1) = \mathrm{St}\partial_t\mathcal{M}_f + v\cdot\nabla_x\mathcal{M}_f$$

Scaled Boltzmann equation

$$Ma\partial_t f + v \cdot \nabla_x f = \frac{1}{Kn} Q(f,f)$$

FAST RELAXATION LIMIT Kn<<1

compressible Euler equations (+ viscous correction O(Kn))

$$Ma\partial_t R + \nabla_x \cdot (RU) = 0$$

$$Ma\partial_t RU + \nabla_x \cdot (RU \otimes U + RT) = O(Kn)$$

$$Ma\partial_t \left(\frac{1}{2}R|U|^2 + \frac{3}{2}RT\right) + \nabla_x \cdot \left(\frac{1}{2}R|U|^2 U + \frac{5}{2}RTU\right) = O(Kn)$$

AROUND A GLOBAL EQUILIBRIUM
INCOMPRESSIBLE LIMIT Ma<<1

Kn<<Ma

incompressible Euler equations

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = 0$$

$$\partial_t \theta + (u \cdot \nabla_x)\theta = 0$$

Kn~Ma

incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = \nu \Delta_x u$$

$$\partial_t \theta + (u \cdot \nabla_x)\theta = \kappa \Delta_x \theta$$

Formal derivation of the incompressible fluid limits

▶ Considering fluctuations around a global equilibrium M

The relative entropy

$$H(f|M) = \iint \left(f \log \frac{f}{M} - f + M \right) dvdx$$

• By the global conservation of mass, momentum and energy, and Boltzmann's H Theorem, it is uniformly bounded

$$H(f|M) \leq H(f_{in}|M)$$

• It is expected to control the size of the fluctuation g defined by

$$f = M(1 + \mathrm{Ma}\, g)$$

Define $h(z) = (1+z)\log(1+z) - z$. Formally

$$H(f|M) = \iint Mh(\mathrm{Ma}\, g) dvdx \sim \frac{1}{2}\mathrm{Ma}^2 \iint Mg^2 dvdx$$

Young's inequality

Assume that

$$H(f_{in}|M) \leq C_{in}\mathrm{Ma}^2.$$

Starting from Young's inequality

$$pz \leq h^*(p) + h(z)\,, \quad \forall p, z \geq 0\,,$$

we get, using the superquadraticity of h^*,

$$\begin{aligned} M|g|(1+|v|^2) &\leq 4\frac{M}{\mathrm{Ma}^2}\left(h(\mathrm{Ma}g) + h^*\left(\frac{\mathrm{Ma}}{4}(1+|v|^2)\right)\right) \\ &\leq 4\frac{M}{\mathrm{Ma}^2}h(\mathrm{Ma}g) + 4Mh^*\left(\frac{1}{4}(1+|v|^2)\right) \end{aligned}$$

meaning that

$$g \in L^\infty_t(L^1_{loc}(dx, L^1(M(1+|v|^2)dv)))$$

▶ Relaxing towards local thermodynamic equilibrium

The scaled Boltzmann equation

$$\mathrm{Ma}\partial_t g + v \cdot \nabla_x g = -\frac{1}{\mathrm{Kn}}\mathcal{L}_M g + \frac{\mathrm{Ma}}{\mathrm{Kn}}\frac{1}{M}Q(Mg, Mg)$$

where $\mathcal{L}_M$ is the linearized collision operator defined by

$$\mathcal{L}_M g = -\frac{2}{M}Q(M, Mg).$$

The thermodynamic constraint

In the limit $\mathrm{Kn} \to 0$, $\mathrm{Ma} \to 0$, we have formally

$$\mathcal{L}_M g = 0$$

(rigorous for instance if g is bounded in some weighted L^2-space)

The linearized collision operator (see [Grad])

• Hilbert's decomposition

$$\mathcal{L}_M g(v) = \nu(|v|)g(v) - \mathcal{K}g(v)$$

where $0 < \nu_- \leq \nu(|v|) \leq \nu_+(1+|v|)^\beta$, and $\mathcal{K}$ is a compact operator (under Grad's cut-off assumption)

• Fredholm alternative

$\mathcal{L}_M$ is a nonnegative unbounded self-adjoint operator on $L^2(Mdv)$ with

$$\mathcal{D}(\mathcal{L}_M) = \{g \in L^2(Mdv) \,|\, \nu g \in L^2(Mdv)\}$$

$$\mathrm{Ker}(\mathcal{L}_M) = \mathrm{Span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

• Coercivity estimate

For each $g \in \mathcal{D}(\mathcal{L}_M) \cap (\mathrm{Ker}(\mathcal{L}_M))^\perp$

$$\int g\mathcal{L}_M g(v)M(v)dv \geq C\|g\|^2_{L^2(M\nu dv)}\,.$$

▶ Deriving the macroscopic constraints

The scaled conservation laws

$$\mathrm{Ma}\partial_t \int Mgdv + \nabla_x \cdot \int vMgdv = 0$$
$$\mathrm{Ma}\partial_t \int vMgdv + \nabla_x \cdot \int v \otimes vMgdv = 0$$
$$\mathrm{Ma}\partial_t \int |v|^2 Mgdv + \nabla_x \cdot \int v|v|^2 Mgdv = 0$$

The macroscopic constraints
In the limit $\mathrm{Kn} \to 0$, $\mathrm{Ma} \to 0$, we have in particular

$$\nabla_x \cdot \int vMgdv = 0$$

$$\nabla_x \cdot \int v \otimes vMgdv = 0$$

In terms of the thermodynamic fields

Plugging the Ansatz

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{|v|^2-3}{2}$$

coming from the thermodynamic constraint, we get

- the incompressibility relation

$$\nabla_x \cdot u = 0;$$

- the Boussinesq relation

$$\nabla_x(\rho+\theta) = 0.$$

Constraints obtained in the zero Mach limit for quasi-homogeneous flows

▶ Taking limits in the evolution equations

The suitable evolution equations

Because of the macroscopic constraints, it is enough to study the asymptotics of the following combinations

$$\partial_t \mathbb{P} \int vMgdv + \frac{1}{\mathrm{Ma}} \mathbb{P}\nabla_x \cdot \int (v \otimes v - \frac{1}{3}|v|^2 Id) Mgdv = 0$$
$$\partial_t \int (|v|^2 - 5) Mgdv + \frac{1}{\mathrm{Ma}} \nabla_x \cdot \int v(|v|^2 - 5) Mgdv = 0$$

Let us define the kinetic fluxes

$$\phi(v) = v \otimes v - \frac{1}{3}|v|^2 Id \in (\mathrm{Ker}\, \mathcal{L}_M)^\perp \Rightarrow \phi = \mathcal{L}_M \tilde{\phi}$$

$$\psi(v) = v(|v|^2 - 5) \in \mathrm{Ker}\, \mathcal{L}_M \Rightarrow \psi = \mathcal{L}_M \tilde{\psi}$$

Because $\mathcal{L}_M$ is self-adjoint and $\mathcal{L}_M g = O(\mathrm{Ma})$, the momentum and enegy fluxes are bounded.

Diffusion and convection terms

The kinetic equation gives

$$\mathcal{L}_M g = \mathrm{Ma}\frac{1}{M}Q(Mg, Mg) - \mathrm{Kn}\, v \cdot \nabla_x g + O(\mathrm{KnMa})$$

Plugging that Ansatz in the fluxes leads to

$$\begin{aligned}\frac{1}{\mathrm{Ma}}\int \zeta Mg dv &= \frac{1}{\mathrm{Ma}}\int \tilde{\zeta} M \mathcal{L}_M g dv \\ &= \underbrace{\int \tilde{\zeta} Q(Mg, Mg) dv}_{\text{convection}} - \underbrace{\frac{\mathrm{Kn}}{\mathrm{Ma}}\int \tilde{\zeta}(v\cdot\nabla_x) Mg dv}_{\text{diffusion}} + O(\mathrm{Kn})\end{aligned}$$

The (formal) limiting equations

In the limit $\mathrm{Kn} \to 0$, $\mathrm{Ma} \to 0$, the thermodynamic constraint gives

$$\partial_t \mathbb{P} u + \mathbb{P}\nabla_x \cdot (u \otimes u) = \left(\lim \frac{\mathrm{Kn}}{\mathrm{Ma}}\right) \mu \Delta_x u,$$

$$\partial_t (3\theta - 2\rho) + 5\nabla_x \cdot (\theta u) = 5\left(\lim \frac{\mathrm{Kn}}{\mathrm{Ma}}\right) \kappa \Delta_x \theta$$

The mathematical difficulties