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Existence and convergence result

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Theorem

For every u_0 with $\phi(u_0) < +\infty$ every family of discrete solutions U_τ admits a subsequence U_{τ_n} uniformly converging to an absolutely continuous function $u : [0, +\infty) \to X$ which solves the metric formulation

$$\frac{d}{dt}\phi(u) = -\|\dot{u}\|\, \|\partial_\ell \phi\|_*(u) = -\psi\big(\|\dot{u}\|\big) - \psi^*\big(\|\partial_\ell \phi\|(u)\big).$$

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If X has the Radon-Nikodym property (e.g. if it is reflexive), then u is a solution of the doubly nonlinear evolution equation

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Time dependent functionals

$$\begin{aligned} \phi(u) \rightsquigarrow \qquad & \phi_{\mathbf{t}}(u) = \phi(u) - \langle \mathbf{f(t), u} \rangle, \quad \mathbf{f} \in C^1(0, +\infty; X') \\ \partial_\ell \phi(u) \rightsquigarrow \qquad & \partial_\ell \phi_{\mathbf{t}}(u) = \partial_\ell \phi(u) - \mathbf{f(t)} \\ & \partial_{\mathbf{t}} \phi_{\mathbf{t}}(u) = -\langle \mathbf{f'(t), u} \rangle \end{aligned}$$

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