2D COMPRESSIBLE VORTEX SHEETS

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EULER'S EQUATIONS OF ISENTROPIC GAS DYNAMICS

We consider a compressible inviscid fluid described by

- the density $\rho(t, x) \in \mathbb{R}$
- the velocity field $\mathbf{u}(t, x) \in \mathbb{R}^d$
- the pressure $p = p(\rho)$, where $p \in \mathcal{C}^\infty$, $p' > 0$,

whose evolution is governed by the Euler equations

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \, \mathbf{u}) = 0, \\ \partial_t (\rho \, \mathbf{u}) + \nabla_x \cdot (\rho \, \mathbf{u} \otimes \mathbf{u}) + \nabla_x \, p(\rho) = 0, \end{cases} \tag{1}$$

where $t \geq 0$ denotes the time variable, $x \in \mathbb{R}^d$ the space variable.

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Smooth solutions

The Euler equations may be written as a symmetric hyperbolic system. This allows to solve locally in time the Cauchy problem:

- Initial data $\rho_0 \in \overline{\rho} + H^s(\mathbb{R}^d)$, $\mathbf{u}_0 \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$. Existence and uniqueness of a solution in the space $C([0,T]; \overline{\rho} + H^s(\mathbb{R}^d)) \times C([0,T]; H^s(\mathbb{R}^d))$ [Kato, 1975]

- Finite time blow-up of smooth solutions [Sideris, 1985] Formation of singularities (shock waves).

- Global (in time) smooth solutions [Serre, 1997] [Grassin, 1998]

Local smooth solution of the initial boundary value problem under the slip boundary condition $\mathbf{u} \cdot \nu = 0$ (characteristic boundary) [Beirao da Veiga, 1981]

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PIECEWISE SMOOTH SOLUTIONS

The function

$$(\rho, \mathbf{u}) := \begin{cases} (\rho^+, \mathbf{u}^+) & \text{if } x_d > \varphi(t, x_1, \ldots, x_{d-1}) \\ (\rho^-, \mathbf{u}^-) & \text{if } x_d < \varphi(t, x_1, \ldots, x_{d-1}), \end{cases}$$

is a weak solution of the Euler equations if
$(\rho^\pm, \mathbf{u}^\pm)$ is a smooth solution on either sides of the interface
$\Sigma := \{x_d = \varphi(t, x_1, \ldots, x_{d-1})\}$ and
it satisfies the Rankine-Hugoniot jump conditions at Σ:

$$\begin{aligned} \partial_t \varphi \, [\rho] - [\rho \mathbf{u} \cdot \nu] &= 0, \\ \partial_t \varphi \, [\rho \mathbf{u}] - [(\rho \mathbf{u} \cdot \nu)\mathbf{u}] - [p]\nu &= 0, \end{aligned} \tag{2}$$

ν is a (space) normal vector to Σ; $[q] := q^+ - q^-$ denotes the jump of q across Σ.
Σ is an unknown of the problem. Free boundary problem !

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- Existence of one uniformly stable shock wave [Blokhin, 1981] [Majda,1983]

- Existence of two uniformly stable shock waves [Métivier, 1986]

- Existence of one rarefaction wave [Alinhac, 1989]

- Existence of sound waves [Métivier, 1991]

- Existence of one small shock wave [Francheteau & Métivier, 2000]

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Compressible vortex sheets

$(\rho, \mathbf{u})$ is a contact discontinuity if the Rankine-Hugoniot conditions (**??**) are satisfied in the form

$$\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu \,,$$
$$p^+ = p^- \,.$$

p monotone gives equivalently

$$\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu \,,$$
$$\rho^+ = \rho^- \,.$$

The front $\Sigma := \{x_d = \varphi(t, x_1, \ldots, x_{d-1})\}$ is *characteristic* with respect to either side.

Density and normal velocity are continuous across the front Σ.

Jump of tangential velocity $\Rightarrow\Rightarrow$ vortex sheet.

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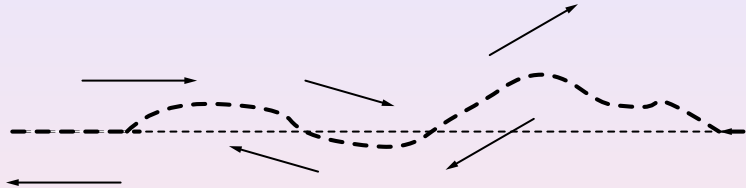
We want to show the (local) existence of compressible vortex sheets (contact discontinuities).

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Linear Spectral Stability (Landau, Miles, ...

Linearize the Euler equations around a *piecewise constant* vortex sheet

$$(\rho, \mathbf{u}) = \begin{cases} (\overline{\rho}, \overline{v}, 0), & \text{if } x_d > 0, \\ (\overline{\rho}, -\overline{v}, 0), & \text{if } x_d < 0. \end{cases}$$

- If $d = 3$, the linearized equations do not satisfy the Lopatinskii condition ($\exists$ exponentially exploding modes!) $\Rightarrow$ violent instability.
- If $d = 2$, and $|[\mathbf{u} \cdot \tau]| < 2\sqrt{2}c(\rho)$ the linearized equations do not satisfy the Lopatinskii condition $\Rightarrow$ violent instability.
- If $d = 2$, and $|[\mathbf{u} \cdot \tau]| > 2\sqrt{2}c(\rho)$ the linearized equations satisfy the weak Lopatinskii condition $\Rightarrow$ weak stability,

where $c(\rho) := \sqrt{p'(\rho)}$ is the sound speed and τ a tangential unit vector to Σ.

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Formulation of the problem

The interface $\Sigma := \{x_2 = \varphi(t, x_1)\}$ is unknown so that the problem is a free boundary problem.

In order to work in a fixed domain $\{y_2 > 0\}$ we introduce the change of variables

$$\begin{aligned}
(\tau, y_1, y_2) &\to (t, x_1, x_2), \\
(t, x_1) &= (\tau, y_1), \\
x_2 &= \Phi(\tau, y_1, y_2),
\end{aligned}$$

where

$$\begin{aligned}
&\Phi : \{(\tau, y_1, y_2) \in \mathbb{R}^3\} \to \mathbb{R}, \\
&\Phi(\tau, y_1, 0) = \varphi(t, x_1), \quad \partial_{y_2}\Phi(\tau, y_1, y_2) \geq \kappa > 0.
\end{aligned}$$

We write again (t, x_1, x_2) instead of (τ, y_1, y_2).

Denote $\Phi^{\pm}(t, x_1, x_2) := \Phi(t, x_1, \pm x_2)$.

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By the Rankine-Hugoniot conditions the boundary matrix of the system of equations is singular at $\{x_2 = 0\}$, i.e. the interface is a characteristic boundary.

The $3 + 3$ equations are <u>not sufficient</u> to determine the unknowns $U^\pm := (\rho^\pm, \mathbf{u}^\pm) = (\rho^\pm, v^\pm, u^\pm)$ and $\Phi^\pm$.

We may prescribe that $\Phi^\pm$ solve in the domain $\{x_2 > 0\}$ the eikonal equations

$$\partial_t \Phi^\pm + v^\pm \partial_{x_1} \Phi^\pm - u^\pm = 0 \,.$$

This choice has the advantage that the boundary matrix of the system for $U^\pm$ has <u>constant rank in the whole domain</u> $\{x_2 \geq 0\}$ (<u>uniformly characteristic boundary</u>).

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We obtain the first order system:

$$\partial_t \rho^+ + v^+ \partial_{x_1} \rho^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} \rho^+}{\partial_{x_2} \Phi^+} + \rho^+ \partial_{x_1} v^+$$

$$+ \rho^+ \frac{\partial_{x_2} u^+}{\partial_{x_2} \Phi^+} - \rho^+ \frac{\partial_{x_1} \Phi^+}{\partial_{x_2} \Phi^+} \partial_{x_2} v^+ = 0 \,,$$

$$\partial_t v^+ + v^+ \partial_{x_1} v^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} v^+}{\partial_{x_2} \Phi^+} + \frac{p'(\rho^+)}{\rho^+} \partial_{x_1} \rho^+$$

$$- \frac{p'(\rho^+)}{\rho^+} \frac{\partial_{x_1} \Phi^+}{\partial_{x_2} \Phi^+} \partial_{x_2} \rho^+ = 0 \,,$$

$$\partial_t u^+ + v^+ \partial_{x_1} u^+ + (u^+ - \partial_t \Phi^+ - v^+ \partial_{x_1} \Phi^+) \frac{\partial_{x_2} u^+}{\partial_{x_2} \Phi^+} + \frac{p'(\rho^+)}{\rho^+} \frac{\partial_{x_2} \rho^+}{\partial_{x_2} \Phi^+} = 0,$$

in the <u>fixed domain</u> $\{x_2 > 0\}$.
$(\rho^-, v^-, u^-, \Phi^-)$ should solve a similar system.

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The boundary conditions are

$$
\begin{aligned}
&\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi\,, \\
&(v^+ - v^-)_{|x_2=0}\, \partial_{x_1}\varphi - (u^+ - u^-)_{|x_2=0} = 0\,, \\
&\partial_t\varphi + v^+_{|x_2=0}\, \partial_{x_1}\varphi - u^+_{|x_2=0} = 0\,, \\
&(\rho^+ - \rho^-)_{|x_2=0} = 0\,, \qquad (t,x) \in [0,T] \times \mathbb{R}^2_+\,,
\end{aligned}
$$

that we rewrite in the compact form as

$$
\begin{aligned}
&\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi\,, \\
&\mathbb{B}(U^+_{|x_2=0}, U^-_{|x_2=0}, \varphi) = 0\,.
\end{aligned}
$$

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We obtain the (non standard) IBVP

$$\partial_t U^\pm + A_1(U^\pm)\partial_{x_1} U^\pm + A_2(U^\pm, \nabla\Phi^\pm)\partial_{x_2} U^\pm = 0\,,$$
$$\partial_t \Phi^\pm + v^\pm \partial_{x_1}\Phi^\pm - u^\pm = 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^2_+,$$

$$\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi\,,$$
$$\mathbb{B}(U^+_{|x_2=0}, U^-_{|x_2=0}, \varphi) = 0\,, \qquad (t,x) \in [0,T] \times \mathbb{R}^2_+,$$

$$(U^\pm, \Phi^\pm)_{|t=0} = (U_0^\pm, \Phi_0^\pm), \qquad x \in \mathbb{R}^2_+.$$

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THEOREM (COULOMBEL, S., 2004, 2008)

Let $d = 2$ and consider a piecewise constant weakly stable vortex sheet. Let $T > 0$ and $m \geq 6$.
Consider initial data $(U_0^\pm, \varphi_0)$ that are perturbations in $H^{m+15/2}(\mathbb{R}_+^2) \times H^{m+8}(\mathbb{R})$ of the piecewise constant vortex sheet. The initial data have compact support and satisfy suitable compatibility conditions.
If the perturbation is sufficiently small, then there exists a unique solution $(U^\pm, \varphi)$ on $[0, T]$ with initial data $(U_0^\pm, \varphi_0)$. The solution belongs to the space $H^m(]0, T[\times \mathbb{R}_+^2) \times H^{m+1}(]0, T[\times \mathbb{R})$.

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THE LINEARIZED PROBLEM

Consider a perturbation of the piecewise constant solution

$$U_{r,l} = \begin{pmatrix} \overline{\rho} \\ \pm\overline{v} \\ 0 \end{pmatrix} + \dot{U}_{r,l}(t,x),$$

$$\Phi_{r,l} = \pm x_2 + \dot{\Phi}_{r,l}(t,x),$$

where $U_{r,l}$, $\Phi_{r,l}$ are linked by the Rankine-Hugoniot conditions, $\dot{U}_{r,l}$ and $\dot{\Phi}_{r,l}$ have compact support, and solve the *eikonal* equations

$$\partial_t \Phi_{r,l} + v_{r,l}\partial_{x_1}\Phi_{r,l} - u_{r,l} = 0. \tag{3}$$

Let us consider the linearized equations around $U_{r,l}, \Phi_{r,l}$:

$$\mathcal{L}(U_{r,l}, \Phi_{r,l})W = f \quad \text{in }]0,T[\times\mathbb{R}^2_+,$$

$$\mathcal{B}(U_{r,l}, \Phi_{r,l})(W, \psi) = g \quad \text{on }]0,T[\times\mathbb{R}.$$

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A PRIORI L^2 ESTIMATE

THEOREM

Let $T > 0$. Assume that
(i) the piecewise constant solution $\overline{U}^{\pm}$ is weakly stable,
*(ii) $(\overline{U}^{\pm} + \dot{U}_{r,l}, \pm x_2 + \dot{\Phi}_{r,l})$ satisfies the Rankine-Hugoniot conditions and the eikonal equations (**??**),*
(iii) the perturbation $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ has compact support and is sufficiently small *in $W^{3,\infty}(]0,T[\times\mathbb{R}^2_+)$.*
Then there exists a solution of the linearized equations that satisfies the a priori estimate:

$$\|W\|^2_{L^2(]0,T[\times\mathbb{R}^2_+)} + \|W^{nc}_{|x_2=0}\|^2_{L^2(]0,T[\times\mathbb{R})} + \|\psi\|^2_{H^1(]0,T[\times\mathbb{R})}$$
$$\leq C\left(\|f\|^2_{L^2(\mathbb{R}_+;H^1(]0,T[\times\mathbb{R}))} + \|g\|^2_{H^1(]0,T[\times\mathbb{R})}\right).$$

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Scheme of the proof

- Microlocal analysis of the paralinearized system associated to the linearized equations.

- Determination of the roots of the Lopatinskii determinant, the poles and the points of non diagonalization of the symbol.

- The singularities of the solution are (micro)localized on bicharacteristic curves propagating from the boundary in the interior domain.

- Despite the loss of regularity, the linearized problem is well-posed in L^2 with source terms in H^1. [Coulombel, 2005]

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- *Paralinearization of the equations.*

  Using the paradifferential calculus (extension of the pseudodifferential calculus which allows a low regularity of the symbols), we substitute in the equations the paradifferential operators (w.r.t. the tangential variables (t, x_1)) and obtain a system of O.D.E. in x_2 with symbols instead of derivatives in (t, x_1).

  This step essentially reduces to the constant coefficient case.

- *Elimination of the front.*

  The projected boundary condition onto a suitable subspace of the frequency space gives an elliptic equation of order one for the front ψ. One obtains an estimate of the form

  $$\|\psi\|^2_{H^1_\gamma(\omega_T)} \leq \quad C \left(\frac{1}{\gamma^2} \|\mathcal{B}(W^{nc}, \psi)\|^2_{H^1_\gamma(\omega_T)} + \|W^{nc}_{|x_2=0}\|^2_{L^2_\gamma(\omega_T)} \right)$$
  $$+ error\ terms\,,$$

  with no loss of regularity with respect to the source terms.

  Thus, it is enough to estimate $W^{nc}_{|x_2=0}$.

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- *Problem with reduced boundary conditions.*

  The projection of the boundary condition onto the orthogonal subspace gives a boundary condition involving only W^{nc}, i.e. without involving ψ. Thus we are left with the (paradifferential version of the) linear problem for W

$$\mathbf{A}_0^r \, \partial_t W^+ + \mathbf{A}_1^r \, \partial_{x_1} W^+ + \mathbf{I}_2 \, \partial_{x_2} W^+ + \mathbf{A}_0^r \, \mathbf{C}^r \, W^+ = F^+ \,, \quad x_2 > 0 \,,$$

$$\mathbf{A}_0^l \, \partial_t W^- + \mathbf{A}_1^l \, \partial_{x_1} W^- + \mathbf{I}_2 \, \partial_{x_2} W^- + \mathbf{A}_0^l \, \mathbf{C}^l \, W^- = F^- \,, \quad x_2 > 0 \,,$$

$$\mathbf{\Pi} \, \widetilde{\mathbf{M}} \, W_{|x_2=0} = \mathbf{\Pi} \, g \,, \qquad\qquad\qquad\qquad x_2 = 0 \,,$$

$$(4)$$

  where diag $(0, 1, 1)$, and $\mathbf{\Pi}$ denotes the suitable projection operator.

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The boundary is characteristic with constant multiplicity.

The problem satisfies a Kreiss-Lopatinski condition in the weak sense and not uniformly. In fact, the Lopatinski determinant associated to the boundary condition vanishes at some points in the frequency space (only simple roots).

The proof of the L^2 energy estimate is based on the construction of a *degenerate Kreiss' symmetrizer*.

In order to explain the main idea, let us consider for simplicity the *linearization around the piecewise constant solution* (constant coefficients case).

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Then, instead of (**??**), we have a problem of the form ($\widehat{W}$ = Fourier transform in (t, x_1))

$$
\begin{aligned}
(\tau \mathcal{A}_0 + i\eta \mathcal{A}_1)\widehat{W} + \mathcal{A}_2 \frac{d\widehat{W}}{dx_2} &= 0, & x_2 &> 0, \\
\beta(\tau, \eta)\widehat{W^{nc}}(0) &= \widehat{h} & x_2 &= 0.
\end{aligned}
\tag{5}
$$

Because of the characteristic boundary, the two first equations do not involve differentiation with respect to the normal variable x_2:

$$
\begin{aligned}
(\tau + iv_r\eta)\widehat{W}_1^+ - ic^2\eta\,\widehat{W}_2^+ + ic^2\eta\,\widehat{W}_3^+ &= 0, \\
(\tau + iv_l\eta)\widehat{W}_1^- - ic^2\eta\,\widehat{W}_2^- + ic^2\eta\,\widehat{W}_3^- &= 0.
\end{aligned}
$$

For Re $\tau > 0$, we obtain an expression for $\widehat{W}_1^+$ and $\widehat{W}_1^-$ that we plug in the other equations.

This operation yields a system of O.D.E. of the form:

$$
\begin{aligned}
\frac{d\widehat{W^{nc}}}{dx_2} &= \mathcal{A}(\tau, \eta)\,\widehat{W^{nc}}, & x_2 &> 0, \\
\beta(\tau, \eta)\widehat{W^{nc}}(0) &= \widehat{h}, & x_2 &= 0.
\end{aligned}
$$

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Nonlinear stability: Nash-Moser iteration

By <u>microlocalization</u>, the analysis is performed in the neighborhood of points (τ, η) of the following type:

1) Points where $\mathcal{A}(\tau, \eta)$ is <u>diagonalizable</u> and the Lopatinskii condition is <u>satisfied</u>.
By using the classical Kreiss' symmetrizer we obtain an L^2 estimate with no loss of derivatives.

2) Points where $\mathcal{A}(\tau, \eta)$ is <u>diagonalizable</u> and the Lopatinskii condition <u>breaks down</u> (the Lopatinskii determinant has simple roots).
We construct a degenerate Kreiss' symmetrizer; this yields an L^2 estimate with loss of one derivative.

3) Points where $\mathcal{A}(\tau, \eta)$ is <u>not diagonalizable</u>. In those points, the Lopatinskii condition is <u>satisfied</u>.

4) <u>Poles</u> of $\mathcal{A}$. At those points, the Lopatinskii condition is <u>satisfied</u>.
We construct a symmetrizer by working on the original system (**??**).

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TAME ESTIMATE IN SOBOLEV NORM

THEOREM

Let $T > 0$ and let $m \geq 3$ be an integer. Assume that (i) the piecewise constant solution $\overline{U}^{\pm}$ is weakly stable,
(ii) $(\overline{U}^{\pm} + \dot{U}_{r,l}, \pm x_2 + \dot{\Phi}_{r,l})$ satisfies the Rankine-Hugoniot conditions and the eikonal equations (??), (iii) the perturbation $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ has compact support and is sufficiently small in $H^6(]0, T[\times \mathbb{R}_+^2)$.
Then the solution of the linearized equations satisfies the a priori estimate:

$$\|W\|_{H^m(]0,T[\times \mathbb{R}_+^2)} + \|W^{nc}_{|x_2=0}\|_{H^m(]0,T[\times \mathbb{R})} + \|\psi\|_{H^{m+1}(]0,T[\times \mathbb{R})}$$
$$\leq C \Big\{ \|f\|_{H^{m+1}(]0,T[\times \mathbb{R}_+^2)} + \|g\|_{H^{m+1}(]0,T[\times \mathbb{R})} +$$
$$+ \Big(\|f\|_{H^4(]0,T[\times \mathbb{R}_+^2)} + \|g\|_{H^4(]0,T[\times \mathbb{R})} \Big) \|(\dot{U}_{r,l}, \dot{\Phi}_{r,l})\|_{H^{m+3}(]0,T[\times \mathbb{R}_+^2)} \Big\}.$$

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SCHEME OF THE PROOF

- Apply the L^2 energy estimates to the tangential derivatives.
- Normal derivatives estimated via the equations and a vorticity equation. No loss of normal regularity inspite of the characteristic boundary. No need to work in spaces with conormal regularity.

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1) Estimate of *tangential derivatives* $\partial_t^h \partial_{x_1}^k W$ and the *front function* ψ by differentiation of the equations along the tangential directions and application of the L^2 energy estimate given in Theorem 2.

2) Estimate of *normal derivatives*.

Consider the original non linear equations. On both sides of the interface the solution is smooth, the interface is a streamline and there is continuity of the normal velocity across the interface; this suggests to estimate the vorticity on either part of the front.

We define the *"linearized vorticity"*

$$\dot{\xi}_\pm := \partial_{x_1}\dot{u}_\pm - \frac{1}{\partial_{x_2}\Phi_{r,l}}\big(\partial_{x_1}\Phi_{r,l}\,\partial_{x_2}\dot{u}_\pm + \partial_{x_2}\dot{v}_\pm\big).$$

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Then

$$\partial_t \dot{\xi}_\pm + v_{r,l}\partial_{x_1}\dot{\xi}_\pm = \partial_{x_1}\mathcal{F}_2^\pm - \frac{1}{\partial_{x_2}\Phi_{r,l}}(\partial_{x_1}\Phi_{r,l}\partial_{x_2}\mathcal{F}_2^\pm + \partial_{x_2}\mathcal{F}_1^\pm)$$
$$+ \Lambda_1^{r,l}\partial_{x_1}\dot{U}_\pm + \Lambda_2^{r,l}\partial_{x_2}\dot{U}_\pm,$$

where

$$\Lambda_{1,2}^{r,l} = \Lambda_{1,2}^{r,l}(\dot{U}_{r,l}, \nabla\dot{U}_{r,l}, \nabla\dot{\Phi}_{r,l}, \nabla^2\dot{\Phi}_{r,l}).$$

An energy argument gives the apriori estimate for $\dot{\xi}_\pm$.

This yields the estimate of the normal derivatives of the characteristic part of the solution.

This allows to obtain the a priori estimate in the standard Sobolev space $H^m(\Omega_T)$. Otherwise we should work in the anisotropic weighted Sobolev space $H_*^m(\Omega_T)$, as for current-vortex sheets in MHD.

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3) Since

$$\partial_{x_2} W_1^{\pm} = \frac{1}{\langle \partial_{x_1} \Phi_{r,l} \rangle^2} \Big\{ \partial_{x_2} \Phi_{r,l} \left(\partial_{x_1} \dot{u}_{\pm} - \dot{\xi}_{\pm} \right)$$
$$-\partial_{x_1} \Phi_{r,l} \left(\partial_{x_2} T_{r,l} W^{\pm} \right)_3 - \left(\partial_{x_2} T_{r,l} W^{\pm} \right)_2 \Big\},$$

we may estimate $\partial_{x_2} W_1^{\pm}$ by the previous steps.
The estimate of _normal derivatives_ $\partial_{x_2} W^{nc}$ of the _noncharacteristic part_ of the solution follows directly from the equations:

$$\mathbf{I}_2 \, \partial_{x_2} W^{\pm} = F^{\pm} - \mathbf{A}_0^{r,l} \, \partial_t W^{\pm} - \mathbf{A}_1^{r,l} \, \partial_{x_1} W^{\pm} - \mathbf{A}_0^{r,l} \, \mathbf{C}^{r,l} \, W^{\pm} \,,$$

since

$$\mathbf{I}_2 := \text{ diag } (0, 1, 1), \ \ W^{nc} := (W_2^+, W_3^+, W_2^-, W_3^-).$$

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NASH-MOSER ITERATION

We use a Nash-Moser iteration where we force the Rankine-Hugoniot jump conditions and the eikonal equations at each step:

- Start from an approximate solution.

- Regularize the coefficients of the linearized equations, force the Rankine-Hugoniot conditions and the eikonal equations.

- Solve the linearized equations, for well chosen source terms.

- Regularize the new coefficients, force the Rankine-Hugoniot conditions and the eikonal equations etc.

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The nonlinear problem

$$\mathcal{L}(V, \Psi) := \mathbb{L}(V + U^a, \Psi + \Phi^a) - \mathbb{L}(U^a, \Phi^a) = f^a \qquad \text{in } \Omega_T,$$

$$\mathcal{E}(V, \Psi) := \partial_t \Psi + (v^a + v)\, \partial_{x_1} \Psi - u + v\, \partial_{x_1} \Phi^a = 0, \qquad \text{in } \Omega_T,$$

$$\mathcal{B}(V, \psi) := \mathbb{B}((V + U^a)_{|_{x_2=0}}, \psi + \varphi^a) = 0, \qquad \text{on } \omega_T, \quad (6)$$

$$\Psi^+_{|_{x_2=0}} = \Psi^-_{|_{x_2=0}} =: \psi, \qquad \text{on } \omega_T.$$

$$V(t, \cdot) = 0, \quad \Psi(t, \cdot) = 0, \quad \psi(t, \cdot) = 0 \qquad \forall t < 0,$$

where

$$V = (\rho, v, u)^T, \quad U^a = (\rho^a, v^a, u^a)^T,$$

$$\begin{cases} f^a := -\mathbb{L}(U^a, \Phi^a), & t > 0, \\ f^a := 0, & t < 0. \end{cases}$$

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The smoothing operators

Theorem (cfr. Hamilton, Francheteau-Métivier)

Let $T > 0$, $\gamma \geq 1$, and let $M \in \mathbb{N}$, with $M \geq 4$. There exists a family $\{S_\theta\}_{\theta \geq 1}$ of operators

$$S_\theta : \quad \mathcal{F}_\gamma^3(\Omega_T) \times \mathcal{F}_\gamma^3(\Omega_T) \longrightarrow \bigcap_{\beta \geq 3} \mathcal{F}_\gamma^\beta(\Omega_T) \times \mathcal{F}_\gamma^\beta(\Omega_T),$$

where $\mathcal{F}_\gamma^s(\Omega_T) := \left\{ u \in H_\gamma^s(\Omega_T) u = 0 \text{ for } t < 0 \right\}$ and a constant $C > 0$ (depending on M), such that

$$\|S_\theta U\|_{H_\gamma^\beta(\Omega_T)} \leq C\, \theta^{(\beta-\alpha)_+} \|U\|_{H_\gamma^\alpha(\Omega_T)}, \qquad \forall\, \alpha, \beta \in \{1, \ldots, M\},$$

$$\|S_\theta U - U\|_{H_\gamma^\beta(\Omega_T)} \leq C\, \theta^{\beta-\alpha} \|U\|_{H_\gamma^\alpha(\Omega_T)}, \qquad 1 \leq \beta \leq \alpha \leq M, \tag{7}$$

$$\left\| \frac{d}{d\theta} S_\theta U \right\|_{H_\gamma^\beta(\Omega_T)} \leq C\, \theta^{\beta-\alpha-1} \|U\|_{H_\gamma^\alpha(\Omega_T)},$$

$\forall\, \alpha, \beta \in \{1, \ldots, M\}$.

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Theorem (continues...)

Moreover, (i) if $U = (u^+, u^-)$ satisfies $u^+ = u^-$ on ω_T, then $S_\theta u^+ = S_\theta u^-$ on ω_T, (ii) the following estimate holds:

$$\|(S_\theta u^+ - S_\theta u^-)_{|x_2=0}\|_{H_\gamma^\beta(\omega_T)} \le C \, \theta^{(\beta+1-\alpha)_+} \|(u^+ - u^-)_{|x_2=0}\|_{H_\gamma^\alpha(\omega_T)},$$

$\forall \, \alpha, \beta \in \{1, \ldots, M\}$.
There is another family of operators, still denoted S_θ, that acts on functions that are defined on the boundary ω_T, and that enjoy the properties (??), with the norms $\| \cdot \|_{H_\gamma^\alpha(\omega_T)}$.

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The Nash-Moser iteration

The iterative scheme starts from $V_0 = 0, \Psi_0 = 0, \psi_0 = 0$.
Assume that V_k, Ψ_k, ψ_k are already given for $k = 1, \ldots, n$ and verify

$$V_k = 0, \quad \Psi_k = 0, \quad \psi_k = 0 \qquad \text{for } t < 0,$$
$$\Psi_k^+ = \Psi_k^- = \psi_k \qquad \text{on } \omega_T, \ k = 1, \ldots, n.$$

Given $\theta_0 \geq 1$, let us set $\theta_n := (\theta_0^2 + n)^{1/2}$ and consider the smoothing operators S_{θ_n}. Let us set

$$V_{n+1} = V_n + \delta V_n, \qquad \Psi_{n+1} = \Psi_n + \delta \Psi_n, \qquad \psi_{n+1} = \psi_n + \delta \psi_n.$$

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We consider the decomposition

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n)$$

$$= \mathbb{L}'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})(\delta V_n, \delta \Psi_n) + e_n' + e_n'' + e_n''',$$

$$\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n)$$

$$= \mathcal{B}'(V_{n+1/2}, \psi_{n+1/2})(\delta V_n, \delta \psi_n) + \tilde{e}_n' + \tilde{e}_n'' + \tilde{e}_n''',$$

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where

$$e'_k := \mathbb{L}(U^a + V_{k+1}, \Phi^a + \Psi_{k+1}) - \mathbb{L}(U^a + V_k, \Phi^a + \Psi_k)$$
$$-\mathbb{L}'(U^a + V_k, \Phi^a + \Psi_k)(\delta V_k, \delta \Psi_k),$$

$$\tilde{e}'_k := \mathcal{B}(V_{k+1}, \psi_{k+1}) - \mathcal{B}(V_k, \psi_k) - \mathcal{B}'(V_k, \psi_k)(\delta V_k, \delta \psi_k)$$

are the "quadratic errors" of Newton's scheme,

$$e''_k := \mathbb{L}'(U^a + V_k, \Phi^a + \Psi_k)(\delta V_k, \delta \Psi_k)$$
$$-\mathbb{L}'(U^a + S_{\theta_k} V_k, \Phi^a + S_{\theta_k} \Psi_k)(\delta V_k, \delta \Psi_k),$$

$$\tilde{e}''_k := \mathcal{B}'(V_k, \psi_k)(\delta V_k, \delta \psi_k) - \mathcal{B}'(S_{\theta_k} V_k, S_{\theta_k} \psi_k)(\delta V_k, \delta \psi_k)$$

are the "first substitution errors" involving the smoothing operators,

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$$e_k''' := \mathbb{L}'(U^a + S_{\theta_k}V_k, \Phi^a + S_{\theta_k}\Psi_k)(\delta V_k, \delta \Psi_k) \\ -\mathbb{L}'(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})(\delta V_k, \delta \Psi_k),$$

$$\tilde{e}_k''' := \mathcal{B}'(S_{\theta_k}V_k, S_{\theta_k}\psi_k)(\delta V_k, \delta \psi_k) - \mathcal{B}'(V_{k+1/2}, \psi_{k+1/2})(\delta V_k, \delta \psi_k)$$

are the _"second substitution errors"_ involving the _smooth modified state_
$V_{n+1/2}$, $\Psi_{n+1/2}$, $\psi_{n+1/2}$ satisfying the Rankine-Hugoniot conditions and the
eikonal equations.

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Introducing the *new unknown*

$$\delta \dot{V}_n := \delta V_n - \delta \Psi_n \frac{\partial_{x_2}(U^a + V_{n+1/2})}{\partial_{x_2}(\Phi^a + \Psi_{n+1/2})}.$$

gives

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) =$$

$$= (L_{n+1/2} + C_{n+1/2})\delta \dot{V}_n + D_{n+1/2}\,\delta \Psi_n + e'_n + e''_n + e'''_n,$$

$$\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n) = \mathbb{B}'_{n+1/2}(\delta \dot{V}_n, \delta \psi_n) + \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n,$$

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where

$$L_{n+1/2} + C_{n+1/2} \quad := L(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \\ + C(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}),$$

$$D_{n+1/2}\, \delta\Psi_n := \frac{\delta\Psi_n}{\partial_{x_2}(\Phi^a + \Psi_{n+1/2})}\, \partial_{x_2}\Big\{ \mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \Big\},$$

$$\mathbb{B}'_{n+1/2} = \mathbb{B}'(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2}).$$

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Let us set

$$e_n := D_{n+1/2}\,\delta\Psi_n + +e'_n + e''_n + e'''_n,$$

$$\tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n.$$

The iteration proceeds as follows.

Given

$$V_0 = 0, \qquad \Psi_0 = 0, \qquad \psi_0 = 0,$$

$$f_0 = S_0 f^a, \qquad g_0 = 0, \qquad E_0 = 0, \qquad \tilde{E}_0 = 0,$$

$$V_1, \ldots, V_n, \qquad \Psi_1, \ldots, \Psi_n, \qquad \psi_1, \ldots, \psi_n,$$

$$f_1, \ldots, f_{n-1}, \qquad g_1, \ldots, g_{n-1},$$

$$e_0, \ldots, e_{n-1}, \qquad \tilde{e}_0, \ldots, \tilde{e}_{n-1},$$

first compute for $n \geq 1$

$$E_n = \sum_{k=0}^{n-1} e_k, \qquad \tilde{E}_n = \sum_{k=0}^{n-1} \tilde{e}_k.$$

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Then compute f_n, g_n from

$$\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = S_{\theta_n} f^a, \qquad \sum_{k=0}^{n} g_k + S_{\theta_n} \tilde{E}_n = 0,$$

and solve the problem

$$(L_{n+1/2} + C_{n+1/2}) \delta \dot{V}_n = f_n \qquad \text{in } \Omega_T,$$

$$\mathbb{B}'_{n+1/2}(\delta \dot{V}_n, \delta \psi_n) = g_n \qquad \text{on } \omega_T,$$

$$\delta \dot{V}_n = 0, \quad \delta \psi_n = 0 \qquad \text{for } t < 0,$$

finding $(\delta \dot{V}_n, \delta \psi_n)$.
Then compute $\delta \Psi_n = (\delta \Psi_n^+, \delta \Psi_n^-)$ from a suitable modification of the
eikonal equations and consequently $\delta V_n, V_{n+1}, \Psi_{n+1}, \psi_{n+1}$.
Finally compute $e_n, \tilde{e}_n$ from

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = f_n + e_n,$$

$$\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n) = g_n + \tilde{e}_n.$$

$$\tag{8}$$

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Adding (**??**) from 0 to N gives

$$\mathcal{L}(V_{N+1}, \Psi_{N+1}) = S_{\theta_N} f^a + (I - S_{\theta_N})E_N + e_N \,,$$

$$\mathcal{B}(V_{N+1}, \psi_{N+1}) = (I - S_{\theta_N})\tilde{E}_N + \tilde{e}_N \,.$$

Because

$$S_{\theta_N} \rightarrow I \quad \text{as } N \rightarrow +\infty$$

$$e_N \rightarrow 0, \quad \tilde{e}_N \rightarrow 0,$$

we formally obtain the resolution of the problem from

$$\mathcal{L}(V_{N+1}, \Psi_{N+1}) \rightarrow f^a, \quad \mathcal{B}(V_{N+1}, \psi_{N+1}) \rightarrow g^a.$$

The rigorous proof of convergence follows from apriori estimates of V_k, Ψ_k, ψ_k proved by induction for every k.

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The existence of weakly stable shock waves

Consider the Euler equations (**??**) in $\mathbb{R}^d$ where $d = 2$ or 3.
Shock waves solutions to (**??**) are <u>smooth solutions on either side</u> of a
hypersurface $\Sigma = \{x_d = \varphi(t, y), t \in [0, T], y \in \mathbb{R}^{d-1}\}$, satisfying at Σ the
<u>Rankine-</u> Hugoniot conditions

$$
\begin{aligned}
&\rho^+ (u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi) = \rho^- (u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi) =: j\,, \\
&j(\mathbf{u}^+ - \mathbf{u}^-) + (p(\rho^+) - p(\rho^-)) \begin{pmatrix} -\nabla_y \varphi \\ 1 \end{pmatrix} = 0\,,
\end{aligned}
\tag{9}
$$

and the <u>Lax' shock inequalities</u> for a 1-shock wave (for example)

$$
j > 0\,, \quad 0 < \frac{u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^+)\sqrt{1 + |\nabla_y \varphi|^2}} < 1 < \frac{u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^-)\sqrt{1 + |\nabla_y \varphi|^2}}\,.
\tag{10}
$$

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Up to Galilean transformations, the <u>planar shock waves</u> have the form

$$(\rho, v, u) = \begin{cases} U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\ U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0, \end{cases} \tag{11}$$

where

$$\rho_r \, u_r = \rho_l \, u_l =: j, \quad j = \sqrt{\rho_r \, \rho_l \, \frac{p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}, \quad 0 < \frac{u_r}{c(\rho_r)} < 1 < \frac{u_l}{c(\rho_l)} \,.$$

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The (linear) stability of planar shock waves:

THEOREM (MAJDA 1983)

The shock wave (**??**) *is uniformly stable if and only if*

$$\frac{u_r^2}{c(\rho_r)^2} \left(\frac{\rho_r}{\rho_l} - 1 \right) < 1 \,.$$

In particular, when p is a convex function of ρ, this inequality always holds.

Majda constructs shock waves that are close to a uniformly stable planar shock.

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When

$$\frac{u_r^2}{c(\rho_r)^2} \left(\frac{\rho_r}{\rho_l} - 1\right) > 1 \,, \tag{12}$$

the planar shock wave (**??**) is only weakly stable.
Coulombel 2004: the linearized problem around a variable coefficients small perturbation of the planar shock (**??**) satisfies an a priori estimate with a loss of one tangential derivative.

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THEOREM (COULOMBEL, S., 2008)

*Consider a planar shock wave (**??**) that satisfies the weak stability condition
(**??**). Let $T > 0$, and let $\mu \in \mathbb{N}$ be sufficiently large. Then there exists an
integer $\tilde{\mu} \geq \mu$, such that if the initial data $(U_0^{\pm}, \varphi_0)$ have the form*

$$U_0^{\pm} = U_{r,l} + \dot{U}_0^{\pm},$$

*with $\dot{U}_0^{\pm} \in H^{\tilde{\mu}+1/2}(\mathbb{R}_+^2)$, $\varphi_0 \in H^{\tilde{\mu}+3/2}(\mathbb{R})$, if they are compatible up to
order $\tilde{\mu} - 1$, have a compact support, and are sufficiently small, then there
exists a solution $U^{\pm} = U_{r,l} + \dot{U}^{\pm}$, $\Phi^{\pm}$, φ to (**??**), (**??**), (**??**), on the time
interval $[0, T]$. This solution satisfies $\dot{U}^{\pm} \in H^{\mu}(]0, T[\times \mathbb{R}^{d-1} \times \mathbb{R}^+)$,
$\varphi \in H^{\mu+1}(]0, T[\times \mathbb{R}^{d-1})$, and $(\dot{U}^{\pm}, \varphi)_{|t=0} = (\dot{U}_0^{\pm}, \varphi_0)$.*

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Subsonic phase transitions in a Van der Waals fluid

Consider the Euler equations (**??**) in $\mathbb{R}^d$ where $d = 2$ or 3.

Model of isothermal liquid/vapor phase transitions in a van der Waals fluid:

$$p(\rho) = \pi(v) := \frac{RT}{v - b} - \frac{a}{v^2}, \quad v := 1/\rho.$$

Phase transition:

smooth solution of (**??**) on either side of a hypersurface $\Sigma = \{x_d = \varphi(t, y)\}$,
that satisfies the Rankine-Hugoniot jump conditions at each point of Σ:

$$
\begin{aligned}
&\rho^+ \left(u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi \right) = \rho^- \left(u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi \right) =: j, \\
&j \left(\mathbf{u}^+ - \mathbf{u}^- \right) + \left(p(\rho^+) - p(\rho^-) \right) \begin{pmatrix} -\nabla_y \varphi \\ 1 \end{pmatrix} = 0, \\
&j > 0, \quad 0 < \frac{u^\pm - v^\pm \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^\pm) \sqrt{1 + |\nabla_y \varphi|^2}} < 1,
\end{aligned}
\tag{13}
$$

(undercompressive shock waves of type 0, Freistühler 1998,
Lax' shock inequalities are not satisfied)

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together with the generalized equal area rule
(capillary admissibility criterion):

$$\int_{v^-}^{v^+} \pi(v)\, dv = \frac{\pi(v^+) + \pi(v^-)}{2}\, (v^+ - v^-).$$ (14)

Consider a planar phase transition

$$(\rho, v, u) = \begin{cases} U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\ U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0, \end{cases}$$ (15)

that satisfies $\rho_r > \rho_M$, $\rho_l < \rho_m$, and the jump conditions

$$\rho_r\, u_r = \rho_l\, u_l =: j\,, \quad j = \sqrt{\rho_r\, \rho_l\, \frac{p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}\,, \quad 0 < \frac{u_r}{c(\rho_r)} < 1 < \frac{u_l}{c(\rho_l)}\,,$$

$$\int_{v_r}^{v_l} \pi(v)\, dv = \frac{p(\rho_r) + p(\rho_l)}{2}\, (v_r - v_l)\,.$$

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THEOREM (BENZONI-GAVAGE, 1998)

There exist planar phase transitions (??), with $\rho_{r,l}$ close enough to $\rho_{M,m}$, and these planar phase transitions are weakly stable. In any case, the uniform Lopatinskii condition is not satisfied.

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SUBSONIC PHASE TRANSITIONS

THEOREM (COULOMBEL, S., 2008)

Consider a planar phase transition (**??**), *as given in Theorem* **??**. *Let* $T > 0$, *and let* $\mu \in \mathbb{N}$ *be sufficiently large. Then there exists an integer* $\tilde{\mu} \geq \mu$, *such that if the initial data* $(U_0^{\pm}, \varphi_0)$ *have the form*

$$U_0^{\pm} = U_{r,l} + \dot{U}_0^{\pm},$$

with $\dot{U}_0^{\pm} \in H^{\tilde{\mu}+1/2}(\mathbb{R}_+^2)$, $\varphi_0 \in H^{\tilde{\mu}+3/2}(\mathbb{R})$, *if they are compatible up to order* $\tilde{\mu} - 1$, *have a compact support, and are sufficiently small, then there exists a solution* $U^{\pm} = U_{r,l} + \dot{U}^{\pm}$, $\Phi^{\pm}$, φ *to* (**??**), (**??**), (**??**) *on the time interval* $[0, T]$. *This solution satisfies* $\dot{U}^{\pm} \in H^{\mu}(]0, T[\times \mathbb{R}^{d-1} \times \mathbb{R}^+)$, $\varphi \in H^{\mu+1}(]0, T[\times \mathbb{R}^{d-1})$, *and* $(\dot{U}^{\pm}, \varphi)_{|t=0} = (\dot{U}_0^{\pm}, \varphi_0)$.