Nonlinear evolution equations with anomalous diffusion

Part I. Lévy operator

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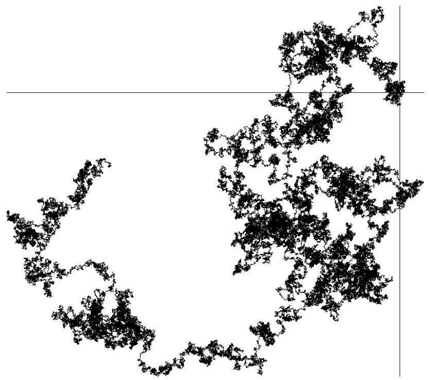
Probabilistic motivations

Laplace operator & Wiener process



Brownian motion – one trajectory of a Wiener process

Laplace operator & Wiener process

Definition
The stochastic process $\{W(t)\}_{t\geq 0}$ is called the Wiener process, if it fulfils the following conditions

- $W(0) = 0$ with probability equal to one,
- $W(t)$ has independent increments ,
- trajectories of W are continuous with probability equal to one
- $\forall_{0\leq s\leq t}\ W_t - W_s \sim \mathcal{N}(0, t-s)$.

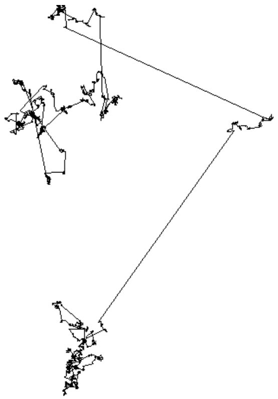
For every function $u_0 \in C_b(\mathbb{R}^n)$ we define

$$u(x,t) = E^x(u_0(W(t))) = \int_{\mathbb{R}^n} u_0(x-y)\, \mathcal{N}(0,t)(dy),$$

where $\mathcal{N}(0,t)(dy) = (2\pi t)^{-n/2} e^{-|x|^2/(2t)}$.
Hence

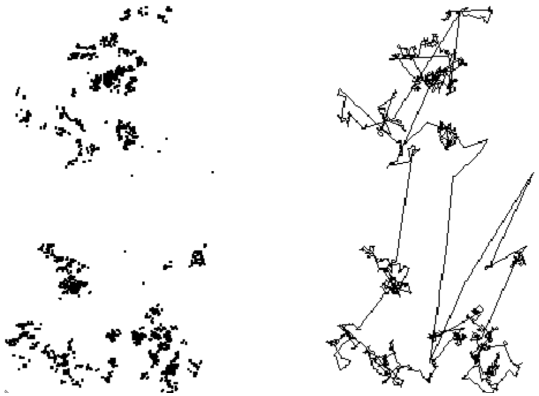
$$u_t = (1/2)\Delta u \quad \text{oraz} \quad u(x,0) = u_0(x).$$

Lévy process



One trajectory of a Lévy process

Lévy process



Two pictures of the same trajectory of a Lévy process

Lévy process

Definition
The stochastic process $\{X(t) \ : \ t \geq 0\}$ on the probability space (Ω, F, P) is called the Lévy process with values in $\mathbb{R}^n$ if it fulfils the following conditions:

- $\mathrm{X}(0) = 0$, P-p.w.,
- for every sequence $0 \leq t_0 < t_1 < \cdots < t_n$ random variables $X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independent,
- the law of $X(s+t) - X(s)$ is independent of s,
- the process $X(t)$ is continuous in probability, namely, $\lim_{s \to t} P(|X_s - X_t| > \varepsilon) = 0$.

Family of measures

We define the family of probability measures

$$\mu^t(dy) = P(X(t) \in dy)$$

and, for every $u_0 \in C_b(\mathbb{R}^n)$,

$$u(x,t) = E^x(u_0(X(t))) = \int_{\mathbb{R}^n} u_0(x-y)\, \mu^t(dy).$$

Convolution semigroup

Definition
The family of bounded Borel measures $\{\mu^t\}_{t\geq 0}$ on $\mathbb{R}^n$ is called to be **the convolution semigroup** if

1. $\mu^t(\mathbb{R}^n) = 1$ for all $t \geq 0$;
2. $\mu^s * \mu^t = \mu^{t+s}$ for $s, t \geq 0$ and $\mu^0 = \delta_0$ (the Dirac delta)
3. $\mu^t \to \delta_0$ vaguely as $t \to 0$, namely,

$$\int_{\mathbb{R}^n} \varphi(y)\, \mu^t(dy) \to \varphi(0) \quad \text{as} \quad t \to 0$$

for every test function $\varphi \in C_c(\mathbb{R}^n)$ (smooth and compactly supported).

Theorem
Let $\{\mu^t\}_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$.
There exists a function $a : \mathbb{R}^n \to \mathbb{C}$ such that

$$\widehat{\mu}^t(\xi) = (2\pi)^{-n/2} e^{-ta(\xi)}$$

holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.

Proof.
For $\xi \in \mathbb{R}^n$ fixed we consider the mapping $\phi_\xi : [0, \infty) \mapsto \mathbb{C}$ defined by

$$\phi_\xi(t) = (2\pi)^{n/2} \widehat{\mu}^t(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\, \mu^t(dx).$$

This mapping is continuous and satisfies

$$\phi_\xi(s+t) = \phi_\xi(t)\phi_\xi(s), \qquad \lim_{t\to 0} \phi_\xi(t) = 1.$$

Hence, there is a unique complex number $a(\xi)$ such that

$$\phi_\xi(t) = e^{-ta(\xi)}, \quad t \geq 0.$$

□

Lévy operator

Definition
Lévy operator $\mathcal{L}$ is the pseudodifferential operator with the symbol $a = a(\xi)$:

$$\widehat{\mathcal{L}v}(\xi) = a(\xi)\widehat{v}(\xi).$$

□

Crucial observation
Denote by $a = a(\xi)$ the symbol of the convolution semigroup $\{\mu^t\}_{t\geq 0}$ in $\mathbb{R}^n$. For every sufficiently regular (bounded) function $u_0 = u_0(x)$ the convolution

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)\, \mu^t(dy).$$

is the solution of the initial value problem

$$u_t = -\mathcal{L}u, \quad x \in \mathbb{R}^n, \quad t \geq 0$$
$$u(x,0) = u_0(x).$$

This is the problem describing **anomalous diffusion**.

Example 1. Let $a(\xi) = |\xi|^2$ and $\mathcal{L} = -\Delta$.
For the heat equation

$$u_t = \Delta u$$

the convolution semigroup $\{\mu^t\}_{t\geq 0}$ has the form

$$\mu^t(dy) = (4\pi t)^{-n/2} e^{-|y|^2/(4t)}\, dy.$$

Hence,

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)(4\pi t)^{-n/2} e^{-|y|^2/(4t)}\, dy.$$

Example 2. With fixed $b \in \mathbb{R}^n$, let $a(\xi) = ib\cdot\xi$ and $\mathcal{L} = b \cdot \nabla$.
In case of the transport equation

$$u_t + b \cdot \nabla u = 0$$

with fixed $b \in \mathbb{R}^n$, we have

$$\mu^t(dx) = \delta_{tb}.$$

Hence,

$$u(x,t) = u_0(x - bt).$$

Theorem (Lévy-Khinchin formula)

There exist

- a vector $b \in \mathbb{R}^n$,
- a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$

$$q(\xi) = \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k,$$

- a Borel measure Π satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} \min(1, |\eta|^2)\, \Pi(d\eta) < \infty$$

such that

$$a(\xi) = ib\cdot\xi + q(\xi) + \int_{\mathbb{R}^n} \left(1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta).$$

Moreover, this representation is unique. □

Lévy operator

Note that

$$\widehat{\mathcal{L}u}(\xi) = a(\xi)\widehat{u}(\xi)$$

with

$$a(\xi) = ib\cdot\xi + q(\xi) + \int_{\mathbb{R}^n} \left(1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta).$$

Inverting the Fourier transform we obtain

$$\begin{aligned}\mathcal{L}u(x) &= b\cdot\nabla u(x) - \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} \\ &\quad - \int_{\mathbb{R}^n} \left(u(x-\eta) - u(x) - \eta\cdot\nabla u(x)\mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta).\end{aligned}$$

Important example: fractional Laplacian

Let

$$\Pi(d\eta) = \frac{C(\alpha)}{|\eta|^{n+\alpha}} \quad \text{with} \quad \alpha \in (0,2)$$

in

$$\mathcal{L}u(x) = -\int_{\mathbb{R}^n} \left(u(x-\eta) - u(x) - \eta\cdot\nabla u(x)\mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta).$$

In this case, we obtain the α-stable anomalous diffusion:

$$\mathcal{L} = (-\Delta)^{\alpha/2} \quad \text{and} \quad a(\xi) = |\xi|^{\alpha} \quad \text{for} \quad 0 < \alpha \le 2.$$

Using symmetry of the Lévy measure, we can simplify:

$$(-\Delta)^{\alpha/2}u(x) = -C(\alpha)\, PV \int_{\mathbb{R}^n} \frac{u(x-\eta) - u(x)}{|\eta|^{n+\alpha}}\, \Pi(d\eta).$$

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Analytical proof:

N. Jacob, *Pseudodifferential Operators and Markov Processes, Vol. 1*, Imperial College Press, 2001.

Probabilistic proof:

J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.

Probabilistic proof of Lévy-Khinchin formula

Definition
We say that a stochastic process $X(t)$ is a Lévy process if for every $s, t \geq 0$, the increment $X_{t+s} - X_t$ is independent of the process $(X_v, 0 \leq v \leq t)$ and has the same law as X_s.

The proof of the Lévy-Khinchin formula consists in showing the decomposition

$$X = X^{(1)} + X^{(2)} + X^{(3)},$$

where

- $X^{(1)}$ is a linear transform of a Brownian motion with drift,
- $X^{(2)}$ is a compound Poisson process having only jumps of size at least 1,
- $X^{(3)}$ is a pure-jump process (martingale) only with jumps of size less than 1.

Maximum principle

Definition
The operator A satisfies the **positive maximum principle** if for any $\varphi \in D(A)$ the fact

$$0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x) \quad \text{for some} \quad x_0 \in \mathbb{R}^n$$

implies

$$A\varphi(x_0) \leq 0.$$

□

REMARK
$A\varphi = \varphi''$ or, more generally, $A\varphi = \Delta\varphi$ satisfies the positive maximum principle.

Maximum principle

Theorem
Denote by $\mathcal{L}$ the Lévy diffusion operator. Then $A = -\mathcal{L}$ satisfies the positive maximum principle.

Proof 1.
Assume that $0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x)$. Then

$$\begin{aligned} & -\mathcal{L}\varphi(x_0) \\ = \; & -b \cdot \nabla\varphi(x_0) + \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 \varphi(x_0)}{\partial x_j \partial x_k} \\ & + \int_{\mathbb{R}^n} \left(\varphi(x_0 - \eta) - \varphi(x_0) - \sum_{j=1}^{n} \eta_j \frac{\partial \varphi(x_0)}{\partial x_j} \mathbb{1}_{\{|\eta|<1\}}(\eta) \right) \Pi(d\eta) \leq 0. \end{aligned}$$

□

Maximum principle

Proof 2.
Assume that $0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x)$.
Recall that the solution of the problem

$$\begin{aligned} & u_t = -\mathcal{L}u, \quad x \in \mathbb{R}^n, \quad t \geq 0, \\ & u(x, 0) = \varphi(x) \end{aligned}$$

is given by

$$u(x, t) = \int_{\mathbb{R}^n} \varphi(x - y) \mu^t(dy).$$

Hence, by the definition of the derivative ∂_t, we have

$$-\mathcal{L}\varphi(x_0) = \lim_{t \to 0^+} \frac{u(x_0, t) - \varphi(x_0)}{t}.$$

Now,

$$u(x_0, t) - \varphi(x_0) = \int_{\mathbb{R}^n} \Big(\varphi(x_0 - y) - \varphi(x_0) \Big) \mu^t(dy) \leq 0.$$

□

Integration by parts and the Lévy operator

Kato inequality for Laplacian

Theorem
For every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta\varphi)\operatorname{sgn}\varphi \, dx \geq 0.$$

Proof. Let

$$g_\varepsilon(s) = \frac{d}{ds}\left(\sqrt{\varepsilon + s^2}\right) = \frac{s}{\sqrt{\varepsilon + s^2}}.$$

Note that

$$g_\varepsilon'(s) \geq 0 \quad \text{and} \quad g_\varepsilon(s) \to \operatorname{sgn} s$$

as $\varepsilon \to 0$. Now, we integrate by parts

$$\int_{\mathbb{R}^n} (-\Delta\varphi)\, g_\varepsilon(\varphi)\, dx = \int_{\mathbb{R}^n} |\nabla\varphi|^2\, g_\varepsilon'(\varphi)\, dx \geq 0,$$

and we pass to the limit $\varepsilon \to 0$.

Kato inequality for Lévy operator

Theorem
For every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (\mathcal{L}\varphi)\operatorname{sgn}\varphi \, dx \geq 0.$$

Proof. Denote by $\{\mu^t\}_{t\geq 0}$ the convolution semigroup corresponding to $\mathcal{L}$. Recall that

$$e^{-t\mathcal{L}}u_0(x) \equiv u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)\,\mu^t(dx)$$

is the solution of the initial value problem

$$u_t = -\mathcal{L}u, \quad x \in \mathbb{R}^n, \;\; t \geq 0$$
$$u(x,0) = u_0(x).$$

Hence

$$\mathcal{L}\varphi = \lim_{t\to 0^+} \frac{\varphi - e^{-t\mathcal{L}}\varphi}{t}.$$

Kato inequality for Lévy operator

Consequently, it suffices to show that

$$\int_{\mathbb{R}^n} (\varphi - e^{-t\mathcal{L}}\varphi)\operatorname{sgn}\varphi \, dx \geq 0$$

which is equivalent to

$$\int_{\mathbb{R}^n} |\varphi| \, dx \geq \int_{\mathbb{R}^n} (e^{-t\mathcal{L}}\varphi)\operatorname{sgn}\varphi \, dx.$$

Now, we complete the proof by the estimate

$$\left|\int_{\mathbb{R}^n} (e^{-t\mathcal{L}}\varphi)\operatorname{sgn}\varphi \, dx\right| \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x-y)|\,\mu^t(dy)\,dx = \int_{\mathbb{R}^n} |\varphi| \, dx.$$

□

Strook-Varopoulos inequality

Theorem
Assume that $\mathcal{L}$ is a Lévy operator.
For every $p \in (1,\infty)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ we have

$$4\frac{p-1}{p^2}\int_{\mathbb{R}^n} (\mathcal{L}\varphi^{p/2})\,\varphi^{p/2}\,dx \leq \int_{\mathbb{R}^n} (\mathcal{L}\varphi)\,\varphi^{p-1}\,dx.$$

REMARK
For $\mathcal{L} = b\cdot\nabla$, both sides of the Strook-Varopoulos inequality are equal to 0.

REMARK
For $\mathcal{L} = -\Delta$ we integrate by parts to obtain **the equality**

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta\varphi)\,\varphi^{p-1}\,dx &= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi|^2\,\varphi^{p-2}\,dx \\ &= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi\,\varphi^{p/2-1}|^2\,dx \\ &= 4\frac{p-1}{p^2}\int_{\mathbb{R}^n} |\nabla\varphi^{p/2}|^2\,dx.\end{aligned}$$

Proof of Strook-Varopoulos inequality

Step 1. Let $\alpha > 0$ and $\beta > 0$ be such that $\alpha + \beta =$. Then

$$(x^\alpha - y^\alpha)(x^\beta - y^\beta) \geq \alpha\beta(x-y)^2$$

for all $x \geq 0$ and $y \geq 0$.

Step 2. We use

$$\int_{\mathbb{R}^n} (\mathcal{L}f)\,g\,dx = \lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^n} (f - e^{-t\mathcal{L}}f)\,g\,dx$$

for all $f, g \in D(\mathcal{L})$.

Step 3. We show (by Step 1) that

$$\int_{\mathbb{R}^n} (f^\alpha - e^{-t\mathcal{L}}f^\alpha)\,f^\beta\,dx \geq \alpha\beta\int_{\mathbb{R}^n} (f - e^{-t\mathcal{L}}f)\,f\,dx$$

for every $f \in D(\mathcal{L})$, $f \geq 0$, and $\alpha + \beta = 2$.

Step 4. We substitute in Step 3

$$f = \varphi^{p/2}, \quad \alpha = \frac{2}{p}, \quad \beta = 2 - \frac{2}{p}, \quad \alpha\beta = 4\frac{p-1}{p^2},$$

and we pass to the limit $t \to 0^+$. □

General Strook-Varopoulos inequality

The Kato inequality combined with the Strook-Varopoulos inequality give the following estimate

$$\frac{4(p-1)}{p^2}\langle \mathcal{L}|\varphi|^{p/2}, |\varphi|^{p/2}\rangle \leq \langle \mathcal{L}\varphi, |\varphi|^{p-1}\mathrm{sgn}\,\varphi\rangle$$

for every $\varphi \in D(\mathcal{L})$.

Convexity inequality

Theorem
Let $u \in C_b^2(\mathbb{R}^n)$ and $g \in C^2(\mathbb{R})$ be a convex function. Then

$$\mathcal{L}g(u) \leq g'(u)\mathcal{L}u.$$

Proof. Use the representation

$$\begin{aligned}\mathcal{L}u(x) &= b\cdot\nabla u(x) - \sum_{j,k=1}^n a_{jk}\frac{\partial^2 u}{\partial x_j\partial x_k} \\ &\quad - \int_{\mathbb{R}^n} \left(u(x-\eta) - u(x) - \eta\cdot\nabla u(x)\mathbb{1}_{\{|\eta|<1\}}(\eta)\right)\,\Pi(d\eta).\end{aligned}$$

and the convexity of g

$$g(u(x-\eta)) - g(u(x)) \geq g'(u(x))[u(x-\eta) - u(x)],$$

which immediately implies

$$g(u(x-\eta)) - g(u(x)) - \eta\cdot\nabla g(u(x)) \geq g'(u(x))[u(x-\eta) - u(x) - \eta\cdot\nabla u(x)].$$

□

Convexity inequality

Corollary
Let $g \in C^2(\mathbb{R})$ be a convex function.
Assume that $g(u) \in D(\mathcal{L})$ and $\mathcal{L}g(u) \in L^1(\mathbb{R}^n)$.
Then

$$0 \left(= \int_{\mathbb{R}^n} \mathcal{L}g(u(x))\, dx \right) \leq \int_{\mathbb{R}^n} g'(u(x))\mathcal{L}u(x)\, dx.$$

Proof. Recall that

$$\int_{\mathbb{R}^n} \mathcal{L}v(x)\, dx = \int_{\mathbb{R}^n} (a\,\widehat{v})\check{}\,(x)\, dx = (2\pi)^{n/2} a(0)\widehat{v}(0)$$

and $a(0) = 0$. □

Important application
Any Lévy diffusion operator $\mathcal{L}$ satisfies

$$\int_{\mathbb{R}^n} (\mathcal{L}u)\Big((u-k)_+\Big)^p\, dx \geq 0$$

for each $1 < p < \infty$ and all constants $k \geq 0$.

Convexity inequality

The General Strook-Varopoulos inequality

$$C(p)\langle \mathcal{L}|\varphi|^{p/2}, |\varphi|^{p/2}\rangle \leq \langle \mathcal{L}\varphi, |\varphi|^{p-1}\operatorname{sgn}\varphi\rangle$$

can be obtained immediatel from the convexity inequality

$$\mathcal{L}g(u) \leq g'(u)\mathcal{L}u.$$

with

$$g(\varphi) = |\varphi|^{p/2} \quad \text{for} \quad p > 2.$$

Here, we have the non-optimal constant

$$C(p) = \frac{2}{p} \quad \left(\leq \frac{4(p-1)}{p^2} \quad \text{for} \quad p > 2 \right).$$

Theorem
The operator $-\mathcal{L}$ generates a strongly continuous semigroup $e^{-t\mathcal{L}}$ of linear operators on $L^2(\mathbb{R})$ (in fact, on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ for a large class of symbols $a(\xi)$).

This is the sub-Markovian semigroup:

$$0 \leq v \leq 1 \quad \text{implies} \quad 0 \leq e^{-t\mathcal{L}} v \leq 1$$

almost everywhere.

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