Nonlinear evolution equations with anomalous diffusion

Part II. Fractal Burgers equation

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Theorem (Lévy-Khinchin formula)

There exist

- a vector $b \in \mathbb{R}^n$,
- a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$

$$q(\xi) = \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k,$$

- a Borel measure Π satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} \min(1, |\eta|^2)\, \Pi(d\eta) < \infty$$

such that

$$a(\xi) = ib\cdot\xi + q(\xi) + \int_{\mathbb{R}^n} \Big(1 - e^{-i\eta\xi} - i\eta\xi 1\!\!1_{\{|\eta|<1\}}(\eta)\Big)\, \Pi(d\eta).$$

Moreover, this representation is unique. □

Lévy operator

Fundamental objects in this lecture:

- Convolution semigroup of measures $\{\mu^t\}_{\geq 0}$.
- Symbol of the convolution semingroup:
$$\widehat{\mu}^t(\xi) = (2\pi)^{-n/2}e^{-ta(\xi)}.$$
- Lévy operator $\mathcal{L}$ is the pseudodifferential operator with the symbol $a = a(\xi)$:
$$\widehat{\mathcal{L}v}(\xi) = a(\xi)\widehat{v}(\xi).$$
- Anomalous diffusion: for every sufficiently regular (bounded) function $u_0 = u_0(x)$ the convolution
$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)\, \mu^t(dy).$$
is the solution of the initial value problem
$$u_t = -\mathcal{L}u, \quad x\in\mathbb{R}^n, \ \ t \geq 0$$
$$u(x,0) = u_0(x).$$

Fractional Laplacian – repetition

Let

$$\Pi(d\eta) = \frac{C(\alpha)}{|\eta|^{n+\alpha}} \quad \text{with} \quad \alpha\in(0,2)$$

in

$$\mathcal{L}u(x) = -\int_{\mathbb{R}^n} \big(u(x-\eta) - u(x) - \eta\cdot\nabla u(x) 1\!\!1_{\{|\eta|<1\}}(\eta)\big)\, \Pi(d\eta).$$

We obtain the α-stable anomalous diffusion:

$$\mathcal{L} = (-\Delta)^{\alpha/2} \quad \text{and} \quad a(\xi) = |\xi|^\alpha \quad \text{for} \quad 0<\alpha<2.$$

The corresponding family of measures satisfies

$$\widehat{\mu}^t(\xi) = (2\pi)^{-n/2}e^{-t|\xi|^\alpha} \quad \text{and} \quad \mu^t(dy) = p_\alpha(y,t)\, dy,$$

where $\widehat{p}_\alpha(\xi,t) = e^{-t|\xi|^\alpha}$.

Fundamental solution

Define the function $p_\alpha(x,t)$ by the Fourier transform:

$$\widehat{p}_\alpha(\xi,t) = e^{-t|\xi|^\alpha}.$$

- Scaling:

$$p_\alpha(x,t) = t^{-n/\alpha} P_\alpha(xt^{-1/\alpha}), \quad \text{where} \quad (P_\alpha)^\vee(\xi) = e^{-|\xi|^\alpha}.$$

- For every $\alpha \in (0,2]$, the function P_α is smooth, nonnegative, $\int_{\mathbb{R}^n} P_\alpha(x)\,dx = 1$, and satisfies

  $$0 \le P_\alpha(x) \le C(1+|x|)^{-(\alpha+n)} \quad \text{and} \quad |\nabla P_\alpha(x)| \le C(1+|x|)^{-(\alpha+n+1)}$$

  for a constant C and all $x \in \mathbb{R}^n$.

Lévy conservation laws

Joint work with P. Biler and W.A. Woyczyński (1999-2001)

Fundamental solution

By the Young inequality for the convolution,

$$S_\alpha(t)v(x) = \left(e^{-t(-\Delta)^{\alpha/2}}\right) v(x) = \int_{\mathbb{R}^n} v(x-y)p_\alpha(y,t)\,dy$$

satisfies

$$\begin{aligned} \|S_\alpha(t)v\|_p &\le Ct^{-n(1-1/p)/\alpha}\|v\|_1, \\ \|\nabla S_\alpha(t)v\|_p &\le Ct^{-n(1-1/p)/\alpha-1/\alpha}\|v\|_1 \end{aligned}$$

for every $p \in [1,\infty]$ and all $t > 0$.

Nonlinear models

Fractal Burgers equation

$$u_t + (-\Delta)^{\alpha/2}u + c\cdot\nabla(u|u|^{r-1}) = 0, \quad c \in \mathbb{R}^n$$

Multifractal conservation laws

$$u_t + \mathcal{L}u + f(u)_x = 0,$$

with the *multifractal operator*

$$\mathcal{L} = -a_0\Delta + \sum_{j=1}^{k} a_j(-\Delta)^{\alpha_j/2},$$

$0 < \alpha_j < 2$, $a_j > 0, j = 0,1,\ldots,k$, where $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian.

Lévy conservation laws: $u_t + \mathcal{L}u + \mathrm{div}\, f(u) = 0$

Existence, uniqueness, and properties of solutions

A priori estimates: $u_t + \mathcal{L}u + \mathrm{div}\, f(u) = 0$

Theorem 2
The solution from Theorem 1 is regular, satisfies the conservation of mass property

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx,$$

and the contraction property

$$\|u(t)\|_p \le \|u_0\|_p,$$

for each $p \in [1,\infty]$ and all $t > 0$. Moreover, the maximum and minimum principles hold true:

$$\mathrm{ess\ inf}\, u_0 \le u(x,t) \le \mathrm{ess\ sup}\, u_0, \quad \text{a.e. } x, t,$$

as well as the comparison principle for $u_0 \le v_0 \in L^1(\mathbb{R}^n)$:

$$u(x,t) \le v(x,t) \ \text{ a.e. } \ x,\ t, \ \text{ and } \ \|u(t) - v(t)\|_1 \le \|u_0 - v_0\|_1.$$

Theorem 1
Let $\mathcal{L} = (-\Delta)^{\alpha/2}$ with $\alpha \in (1,2)$. Assume that $f \in C^1(\mathbb{R},\mathbb{R}^n)$.
Given

$$u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

there exists the unique solution $u \in \mathcal{C}([0,\infty);\ L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of the problem

$$u_t + \mathcal{L}u + \nabla \cdot f(u) = 0, \qquad u(x,0) = u_0(x).$$

Proof.

- Local-in-time (mild) solutions are obtained *via* the integral equation

$$u(t) = S_\alpha(t)u_0 - \int_0^t \nabla \cdot S_\alpha(t-\tau) f(u)(\tau)\,d\tau.$$

- These solutions exist for all $t > 0$ because of *a priori* estimates.

A priori estimates: $u_t + \mathcal{L}u + \mathrm{div}\, f(u) = 0$

For $1 < p < \infty$ we multiply equation by $|u|^{p-1}\mathrm{sgn}\, u$ and integrate over $\mathbb{R}^n$ with respect to x. This leads to the equality

$$\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p\,dx + \langle \mathcal{L}u, |u|^{p-1}\mathrm{sgn}\, u\rangle = 0.$$

The second term on the left hand side of the above formula is nonnegative in view of the Strook-Varopoulos inequality

$$\frac{4(p-1)}{p^2}\langle \mathcal{L}|u|^{p/2}, |u|^{p/2}\rangle \le \langle \mathcal{L}u, |u|^{p-1}\mathrm{sgn}\, u\rangle,$$

and because of the following well-known property of the Fourier transform

$$\langle \mathcal{L}\varphi, \varphi\rangle = \langle \widehat{\mathcal{L}\varphi}, \widehat{\varphi}\rangle = \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{\varphi}(\xi)|^2\,d\xi \ge 0.$$

Comparison principle: $u_t + \mathcal{L}u + \mathrm{div}\, f(u) = 0$

Now, we show that solutions of the Lévy conservation law with $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy

$$\mathrm{ess\ inf}\, u_0 \le u(x,t) \le \mathrm{ess\ sup}\, u_0, \quad \text{a.e. } x,\ t.$$

Let $k = \mathrm{ess\ sup}\, u_0$ and consider the function

$$g = (u-k)_+ \equiv \max(u-k, 0).$$

Multiplying the equation by g we obtain

$$\int_{\mathbb{R}^n} g u_t \, dx + \int_{\mathbb{R}^n} g\, \mathcal{L}u \, dx + \int_{\mathbb{R}^n} g \nabla \cdot f(u) \, dx = 0.$$

Since $\int_{\mathbb{R}^n} u_t g \, dx = \int_{\mathbb{R}^n} g_t g \, dx$, $\int_{\mathbb{R}^n} g \nabla \cdot f(u) \, dx = 0$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} g^2(x,t)\, dx + \langle g, \mathcal{L}u \rangle = 0.$$

The convexity inequality implies $\langle \mathcal{L}u, g \rangle \ge 0$. Hence

$$g \equiv 0.$$

Self-similar large time behavior – linear asymptotics

Theorem A (Biler, K., Woyczyński (2001))
Assume that $\alpha \in (1,2)$ and $q > 0$.
Let u be the solution of the Cauchy problem

$$u_t + (-\Delta)^{\alpha/2} u + b \cdot \nabla \left(u|u|^q\right) = 0, \quad u(x,0) = u_0(x).$$

Suppose that the initial datum satisfies

$$u_0 \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} u_0(x)\, dx = M$$

for some fixed $M \in \mathbb{R}$.

- If $q > (\alpha-1)/n$, then then

$$t^{n(1-1/p)/\alpha} \| u(t) - M p_\alpha(t) \|_p \to 0 \quad as \quad t \to \infty,$$

for each $p \in [1,\infty]$.

Ref.: [P. Biler, G. K., and W.A. Woyczyński, *Asymptotics for conservation laws involving Levy diffusion generators*, Studia Math. **148** (2001), 171–192.]

Large time behavior of solutions for $u_0 \in L^1(\mathbb{R}^n)$.

Self-similar large time behavior – nonlinear asyptotics

Theorem B (Biler, K., Woyczyński, (2001))
Assume that $\alpha \in (1,2)$ and $q > 0$.
Let u be the solution of the Cauchy problem

$$u_t + (-\Delta)^{\alpha/2} u + b \cdot \nabla \left(u|u|^q\right) = 0, \quad u(x,0) = u_0(x).$$

Suppose that $u_0 \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} u_0(x)\, dx = M$.

- If $q = (\alpha-1)/n$, then

$$t^{n(1-1/p)/\alpha} \| u(t) - U_M(t) \|_p \to 0 \quad as \quad t \to \infty,$$

for each $p \in [1,\infty]$, where $U_M(x,t) = t^{-n/\alpha} U_M(xt^{-1/\alpha}, 1)$ is the unique self-similar solution of the equation

$$u_t + (-\Delta)^{\alpha/2} u + b \cdot \nabla (u|u|^{(\alpha-1)/n}) = 0$$

with the initial datum $M\delta_0$.

Ref.: P. Biler, G. K., and W.A. Woyczyński, *Critical nonlinearity exponent and self-similar asymptotics for Levy conservation laws* , Ann. I.H. Poincaré - Analyse non linéare **18**, (2001), 613-637.]

Fractal Burgers equation

Joint work with C. Miao and X. Xu (2007)
and with C. Imbert (2008).

$1 < \alpha < 2$. Existence o solutions

Theorem (Biler, K., Woyczyński, Droniou, Imbert, Gallouët, Vovelle)

Let $\alpha \in (1, 2)$ and $u_0 \in L^\infty(\mathbb{R})$.

There exists the unique smooth global-in-time solution $u = u(x,t)$ to the Cauchy problem

$$\begin{aligned} &u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x). \end{aligned}$$

Moreover, the following inequality holds true

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \quad \text{for all} \quad t > 0.$$

Initial value problem

The Cauchy problem

$$\begin{aligned} &u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &u(0,x) = u_0(x) \end{aligned}$$

where

$$\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$$

is defined *via* the Fourier transform $\widehat{(\Lambda^\alpha v)}(\xi) = |\xi|^\alpha \widehat{v}(\xi)$.

Initial condition

$$u_0(x) = c + \int_{-\infty}^x m(dy)$$

with $c \in \mathbb{R}$, m being a finite signed measure on $\mathbb{R}$.

REMARK
For $c = 0$ and a probability measure m, the function u_0 is called the **probability distribution function**.

$1 < \alpha < 2$. Decay estimates

Theorem (K., Miao, Xu)

Let $\alpha \in (1, 2)$ and

$$u_0(x) = c + \int_{-\infty}^x m(dy)$$

with $c \in \mathbb{R}$ and m being a finite **nonnegative** measure on $\mathbb{R}$.

Then

- $u_x(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$,
- $\int_\mathbb{R} u_x(x,t)\, dx = \int_\mathbb{R} m(dx)$,
- for every $p \in [1, \infty]$

$$\|u_x(t)\|_p \leq t^{-1+1/p} \|m\|^{1/p}.$$

Nonnegativity of $u_x(x,t)$

Assume that $u_{0,x}(x) \geq 0$ and $u_{0,x} \in L^p(\mathbb{R})$ for every $p \in [1,\infty]$. Differentiating equation with respect to x we have

$$(u_x)_t + \Lambda^\alpha u_x + (uu_x)_x = 0.$$

Hence, multiplying equation by u_x^-, integrating over $\mathbb{R}$, and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{u_x\leq 0}(u_x^-)^2\,dx + \int_{\mathbb{R}}(\Lambda^\alpha u_x)u_x^-\,dx &= -\int_{u_x\leq 0}(uu_x^-)_x u_x^-\,dx\\ &= -\frac{1}{2}\int_{u_x\leq 0}(u_x^-)^3\,dx.\end{aligned}$$

Since $\int_{\mathbb{R}}(\Lambda^\alpha u_x)u_x^-\,dx \geq 0$ and $\int_{u_x\leq 0}(u_x^-(x,0))^2\,dx = 0$, we have $\int_{u_x\leq 0}(u_x^-(x,t))^2\,dx = 0$ for all $t \geq 0$. Consequently,

$$u_x^-(x,t) \equiv 0.$$

Inequality $\|u_x(t)\|_p \leq t^{-1+1/p}\|m\|^{1/p}$

For fixed $p \in (1,\infty)$ and $u_x \geq 0$, we multiply the equation

$$(u_x)_t + \Lambda^\alpha u_x + (uu_x)_x = 0$$

by u_x^{p-1} and integrate over $\mathbb{R}$:

$$\frac{1}{p}\frac{d}{dt}\|u_x\|_p^p + \int_{\mathbb{R}} u_x^{p-1}\Lambda^\alpha u_x\,dx = -\int_{\mathbb{R}}(uu_x)_x u_x^{p-1}\,dx = -\frac{p-1}{p}\int_{\mathbb{R}} u_x^{p+1}\,dx.$$

Recall now that $\int_{\mathbb{R}} u_x^{p-1}\Lambda^\alpha u_x\,dx \geq 0$ by the convexity inequality.
It follows from the Hölder inequality that

$$\|u_x(t)\|_p^{p^2/(p-1)} \leq \|u_x(t)\|_{p+1}^{p+1}\|m\|^{1/(p-1)}.$$

Hence, (note that $u_x \geq 0$) we obtain the following differential inequality

$$\frac{d}{dt}\|u_x(t)\|_p^p \leq -(p-1)\|m\|^{-1/(p-1)}\left(\|u_x(t)\|_p^p\right)^{p/(p-1)}.$$

Integrating it we complete the proof for any $p \in (1,\infty)$.
The case of $p=\infty$ is obtained passing to the limit $p \to \infty$.

Conservation of the integral

We consider

$$u_t + \Lambda^\alpha u + uu_x = 0 \quad \text{for} \quad \alpha \in (1,2),$$

with the initial datum

$$u_0(x) = c + \int_{-\infty}^x m(dy)$$

with $c \in \mathbb{R}$ and m being a finite **nonnegative** measure on $\mathbb{R}$.
Consider integral equation for u_x:

$$u_x(t) = S_\alpha(t)m - \int_0^t \partial_x S_\alpha(t-\tau)u(\tau)u_x(\tau)\,d\tau.$$

Integrating equation over $\mathbb{R}$ and using the equalities

$$\int_{\mathbb{R}} S_\alpha(t)m\,dt = \int_{\mathbb{R}} m(dx) \quad \text{and} \quad \int_{\mathbb{R}} \partial_x S_\alpha(t-\tau)(u(\tau)u_x(\tau))\,dx = 0$$

we obtain the identity

$$\int_{\mathbb{R}} u_x(x,t)\,dx = \int_{\mathbb{R}} m(dx) \equiv \|m\|.$$

Rarefaction waves

The unique entropy solution of the Riemann problem

$$\begin{aligned} &w_t^R + w^R w_x^R = 0,\\ &w^R(x,0) = w_0^R(x) = \begin{cases} u_-\,, & x<0,\\ u_+\,, & x>0,\end{cases}\end{aligned}$$

with $u_- < u_+$, is given by the so-called **rarefaction wave**

$$w^R(x,t) = W^R(x/t) = \begin{cases} u_-\,, & x/t \leq u_-,\\ x/t\,, & u_- \leq x/t \leq u_+,\\ u_+\,, & x/t \geq u_+.\end{cases}$$

$1 < \alpha < 2$. Large time asymptotics

Theorem (K., Miao, Xu)

Let $\alpha \in (1, 2)$ and
$$u_0(x) = c + \int_{-\infty}^{x} m(dy)$$
with $c \in \mathbb{R}$ and m being a finite measure on $\mathbb{R}$ **(not necessary nonnegative)**. We assume that
$$u_- = c < u_+ = c + \int_{-\infty}^{+\infty} m(dy).$$
For every
$$p \in \left(\frac{3-\alpha}{\alpha-1}, \infty\right]$$
there exists $C > 0$ independent of t such that
$$\|u(t) - w^R(t)\|_p \le Ct^{-[\alpha-1-(3-\alpha)/p]/2} \log(2+t)$$
for all $t > 0$.

Smooth approximations of rarefaction waves

Theorem
Let $u_- < u_+$. The problem
$$w_t - w_{xx} + ww_x = 0,$$
$$w(x,0) = w_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$
has the unique, smooth, global-in-time solution $w(x,t)$ satisfying

- $u_- < w(t,x) < u_+$ and $w_x(t,x) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$;
- for every $p \in [1, \infty]$, there exists a constant $C = C(p, u_-, u_+) > 0$ such that
$$\begin{aligned} \|w_x(t)\|_p &\le Ct^{-1+1/p}, \\ \|w_{xx}(t)\|_p &\le Ct^{-3/2+1/(2p)} \\ \|w(t) - w^R(t)\|_p &\le Ct^{-(1-1/p)/2}, \end{aligned}$$
for all $t > 0$, where $w^R(x,t)$ is the rarefaction wave.

This result is deduced from the explicit formula for smooth approximations of rarefaction waves.

Smooth approximation of the rarefaction wave

In the proof, we use the "energy estimates" in order to show the convergence toward **the smooth approximation of the rarefaction wave**, i.e., the unique smooth solution to the viscous Burgers equation
$$w_t - w_{xx} + ww_x = 0,$$
$$w(x,0) = w_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$

Proof of convergence toward rarefaction waves

Using the Gagliardo-Nirenberg inequality we have
$$\|u(t) - w(t)\|_p \le C\Big(\|u_x(t)\|_\infty + \|w_x(t)\|_\infty\Big)^a \|u(t) - w(t)\|_{p_0}^{1-a}$$
for $1 < p_0 < p < \infty$.

Since $\|u_x(t)\|_\infty$ and $\|w_x(t)\|_\infty$ decay, the proof is completed by the following lemma.

Lemma
For $p_0 = (3-\alpha)/(\alpha-1)$, the following estimate is valid
$$\|u(t) - w(t)\|_{p_0} \le C \log(2+t).$$

Proof of Lemma.

The function $v = u - w$ satisfies

$$v_t + \Lambda^\alpha v + \frac{1}{2}[v^2 + 2vw]_x = -\Lambda^\alpha w + w_{xx}.$$

We multiply this equation by $|v|^{p-2}v$ and we integrate over $\mathbb{R}$ to obtain

$$\frac{1}{p}\frac{d}{dt}\int |v|^p\, dx + \int (\Lambda^\alpha v)(|v|^{p-2}v)\, dx + \frac{1}{2}\int [v^2 + 2vw]_x |v|^{p-2} v\, dx$$
$$= \int (-\Lambda^\alpha w + w_{xx})(|v|^{p-2}v)\, dx.$$

The second and the third term on the left hand side are nonnegative. Using the Hölder inequality on the right-hand side we obtain the following differential inequality

$$\frac{d}{dt}\|v(t)\|_p^p \leq p\left(\|\Lambda^\alpha w(t)\|_p + \|w_{xx}(t)\|_p\right)\|v(t)\|_p^{p-1},$$

which, after integration, leads to

$$\|v(t)\|_p \leq \|v(t_0)\|_p + \int_{t_0}^{t} \|\Lambda^\alpha w(\tau)\|_p + \|w_{xx}(\tau)\|_p\, d\tau.$$

□

Publicity

From: http://wikitravel.org/en/Wroclaw

Wroclaw in Polish, formally known as **Breslau** in German, is a large undiscovered gem of a city in southwestern Poland in the historic region of Silesia. It boasts fascinating architecture, many rivers and bridges, and a lively and metropolitan cultural scene. It is a city with a troubled past, having seen much violence and devastation, and was almost completely destroyed during the end of the Second World War. However, it has been brilliantly restored and can now be counted amongst the highlights of Poland, and all of Central Europe. As Poland rushes headlong into further integration with the rest of Europe, now is the time to visit before the tourist hordes (and high prices) arrive. Read Norman Davies'and Roger Moorhouse's Microcosm: Portrait of a Central European City to understand the complicated history of the town.

Publicity

Entropy solutions for $\alpha \in (0, 1]$

Entropy solutions for $0 < \alpha \leq 1$

Theorem (Alibaud)

Let $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$.

There exists the unique **entropy** (in the sense of Kruzhkov) global-in-time solution $u = u(x,t)$ to the Cauchy problem

$$\begin{aligned} &u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x). \end{aligned}$$

Proof of decay estimates for $p = 2$

For $v = u_x^\varepsilon \geq 0$, we multiply the equation

$$v_t - \varepsilon v_{xx} + \Lambda^\alpha v + (u^\varepsilon v)_x = 0$$

by v and integrate over $\mathbb{R}$:

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \varepsilon \int_{\mathbb{R}} (v_x)^2\,dx + \int_{\mathbb{R}} v\Lambda^\alpha v\,dx + \frac{1}{2}\int_{\mathbb{R}} v^3\,dx = 0.$$

Note that second, third, and forth term are nonnegative !

Let us use the third one, only.

$0 < \alpha \leq 2$. Decay estimates

Theorem
Let $0 < \alpha \leq 2$ and $u_0(x) = c + \int_{-\infty}^x m(dy)$ with $c \in \mathbb{R}$ and m being a finite **nonnegative** measure on $\mathbb{R}$.

For any $\varepsilon > 0$, denote by $u^\varepsilon = u^\varepsilon(x,t)$ the unique solutions of the regularized problem

$$\begin{aligned} &u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + u^\varepsilon u_x^\varepsilon = 0, \quad x \in \mathbb{R},\ t > 0, \\ &u^\varepsilon(x,0) = u_0(x). \end{aligned}$$

Then

- $u_x^\varepsilon(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$,
- for every $p \in [1,\infty]$ there exists $C = C(p) > 0$ **independent** of ε such that

  $$\|u_x^\varepsilon(t)\|_p \leq C(p) \min\left\{ t^{-(1/\alpha)(1-1/p)}\|m\|, t^{-(1-1/p)}\|m\|^{1/p} \right\}$$

  for all $t > 0$

Nash inequality for the operator Λ^α

Lemma
Let $0 < \alpha$. There exists a constant $C_N > 0$ such that

$$\|w\|_2^{2(1+\alpha)} \leq C_N \langle \Lambda^\alpha w, w\rangle \|w\|_1^{2\alpha}$$

for all functions w satisfying $w \in L^1(\mathbb{R})$ and $\Lambda^{\alpha/2} w \in L^2(\mathbb{R})$.

Proof.
For every $R > 0$, we decompose the L^2-norm of the Fourier transform of w as follows

$$\begin{aligned} \|w\|_2^2 &= C\int_{\mathbb{R}} |\widehat{w}(\xi)|^2\,d\xi \\ &\leq C\|\widehat{w}\|_\infty^2 \int_{|\xi|\leq R} d\xi + CR^{-\alpha}\int_{|\xi|>R} |\xi|^\alpha |\widehat{w}(\xi)|^2\,d\xi \\ &\leq CR\|w\|_1^2 + CR^{-\alpha}\|\Lambda^{\alpha/2}w\|_2^2. \end{aligned}$$

For $R = \left(\|\Lambda^{\alpha/2}w\|_2^2/\|w\|_1^2\right)^{1/(1+\alpha)}$ we obtain complete the proof. $\square$

End of the proof

Applying the Nash inequality, we estimate the third term of the inequality

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \varepsilon \int_{\mathbb{R}} (v_x)^2\,dx + \int_{\mathbb{R}} v\Lambda^\alpha v\,dx + \frac{1}{2}\int_{\mathbb{R}} v^3\,dx = 0.$$

to obtain

$$\frac{d}{dt}\|v(t)\|_2^2 + 2C_N^{-1}\|m\|^{-2\alpha}\|v(t)\|_2^{2(1+\alpha)} \le 0,$$

which, after integration, leads to

$$\|v(t)\|_2 \le C_1\|m\|t^{-1/(2\alpha)} \quad \text{with} \quad C_1 = (C_N/2\alpha)^{1/(2\alpha)}.$$

This is requived decay estimate with $p = 2$. □

In fact, we have

$$\|u_x^\varepsilon(t)\|_p \le C(p)\min\left\{t^{-(1/\alpha)(1-1/p)}\|m\|,\ t^{-(1-1/p)}\|m\|^{1/p}\right\}$$

$0 < \alpha < 1$. Regularized problem ($\varepsilon > 0$ is fixed)

LEMMA
Let $0 < \alpha \le 1$ and $u_0(x) = c + \int_{-\infty}^x m(dy)$ with $c \in \mathbb{R}$ and m being a finite measure on $\mathbb{R}$ **(not necessary nonnegative)**.
Denote $S_\alpha^\varepsilon(t) = e^{-t\Lambda^\alpha + \varepsilon t\partial_x^2}$.

Then, for every $p \in [1,\infty]$ there exists $C(p) > 0$ **independent** of ε such that

$$\begin{aligned}\|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p &\le \int_0^t \left\|S_\alpha^\varepsilon(t-\tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\right\|_p\,d\tau \\ &\le C_p\|u_0\|_\infty\|m\|t^{1-(1/\alpha)(1-1/p)}\end{aligned}$$

for all $t > 0$.

$0 < \alpha < 1$ in equation $u_t + \Lambda^\alpha u + uu_x = 0$

Passing to the limit $\varepsilon \to 0$ in the estimate from the lemma above we obtain

Theorem
Let $0 < \alpha < 1$ and $u_0(x) = c + \int_{-\infty}^x m(dy)$ with $c \in \mathbb{R}$ and m being a finite **nonnegative** measure on $\mathbb{R}$. Put $M = \int_{\mathbb{R}} m(dx)$.

Denote $S_\alpha(t) = e^{-t\Lambda^\alpha}$.

Let $u = u(x,t)$ be the entropy solutions to the fractal Burgers equation.

Then, for every $p \in \left(\frac{1}{1-\alpha}, \infty\right]$ there exists $C(p) > 0$ such that

$$\|u(t) - Mp_\alpha(t)\|_p \quad \le \quad C_p\|u_0\|_\infty\|m\|t^{1-(1/\alpha)(1-1/p)}$$

for all $t > 0$.

$0 < \alpha < 2$. Asymptotic stability

LEMMA (K., Miao, Xu)
Let $\alpha \in (0,2)$. Assume that u^ε and $\widetilde{u^\varepsilon}$ are two solutions of the regularized problem

$$\begin{aligned}&u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + u^\varepsilon u_x^\varepsilon = 0, \quad x \in \mathbb{R},\ t > 0,\\ &u^\varepsilon(x,0) = u_0(x).\end{aligned}$$

with initial conditions u_0 and $\widetilde{u}_0$, the both of with finite signed measures m and $\widetilde{m}$, respectively.

Suppose, moreover, that the measure $\widetilde{m}$ of $\widetilde{u}_0$ is nonnegative and $u_0 - \widetilde{u}_0 \in L^1(\mathbb{R})$.

Then, for every $p \in [1,\infty]$ there exists a constant $C = C(p) > 0$ **independent** of ε such that

$$\|u^\varepsilon(t) - \widetilde{u^\varepsilon}(t)\|_p \le Ct^{-(1-1/p)/\alpha}\|u_0 - \widetilde{u}_0\|_1$$

for all $t > 0$.

$\alpha = 1$. Self-similar solution

Theorem
The unique entropy solution $U = U(x,t)$ of the initial value problem

$$U_t + \Lambda^1 U + UU_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$U(x,0) = U_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases}$$

is self-similar, i.e. it has the form

$$U(x,t) = U\left(\frac{x}{t}, 1\right)$$

for all $x \in R$ and $t > 0$.

Proof. It follows immediately from the Alibaud uniqueness result, because the problem is invariant under the rescaling

$$U^\lambda(x,t) = U(\lambda x, \lambda t).$$

Probabilistic conclusion

Solutions of the initial value problem

$$u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(0,x) = u_0(x)$$

where

$$u_0(x) = \int_{-\infty}^{x} m(dy),$$

with a probability measure m on $\mathbb{R}$, converge, as $t \to \infty$, toward

- the uniform distribution on the interval $[0,t]$, if $1 < \alpha \le 2$;
- the one parameter family of new laws, if $\alpha = 1$;
- the symmetric α-stable law, if $0 < \alpha < 1$.

$\alpha = 1$. Large time asymptotics

Theorem
Let $u = u(x,t)$ be the entropy solution to $u_t + \Lambda^1 u + uu_x = 0$ corresponding to the initial condition $u_0(x) = c + \int_{-\infty}^{x} m(dy)$ with $c \in \mathbb{R}$ and m being a finite measure on $\mathbb{R}$ **(not necessary nonnegative)**.

We assume that $u_- = c < u_+ = c + \int_{-\infty}^{+\infty} m(dy)$ and $u_0 - U_0 \in L^1(\mathbb{R})$.

For every $p \in [1,\infty]$ there exists $C > 0$ independent of t such that

$$\|u(t) - U(t)\|_p \le Ct^{-(1-1/p)}\|u_0 - U_0\|_1$$

for all $t > 0$.

Proof. Pass to the limit $\varepsilon \to 0$ in Asymptotic stability lemma.

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