Nonlinear evolution equations with anomalous diffusion

Part III.

Fractal Hamilton-Jacobi-KPZ equations

Grzegorz Karch

Wrocław, POLAND
http://www.math.uni.wroc.pl/∼karch/

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PHYSICAL MOTIVATION

- A surface is grown via a ballistic deposition process (such as some chemical vapor deposition processes in semiconductor growth).
- New particles are added in the direction perpendicular to the existing surface.

Denote by $h(x,t)$ the function describing the evolution of the interface elevation. Since the normal vector is

$$n = \frac{(-\nabla_x h, 1)}{\sqrt{1+(\nabla_x h)^2}},$$

the elevation increment satisfies

$$\delta h = v\frac{1}{\sqrt{1+|\nabla_x h\|^2}} \cdot \delta t \approx \left(v - \frac{v}{2}|\nabla_x h|^2\right)\delta t.$$

Here, v stands for the velocity of particles being deposited.
Taking the limit $\delta t \to 0$, and transforming to another coordinate frame, we obtain the Hamilton-Jacobi equation

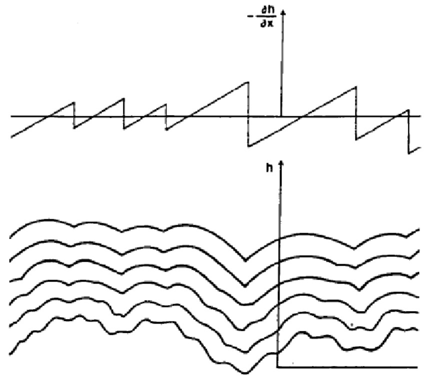
$$h_t + \lambda|\nabla_x h|^2 = 0,$$

where $\lambda \in \mathbb{R}$ is constant.

Hamilton-Jacobi equation $h_t + \lambda|\nabla_x h|^2 = 0$.

In the one dimensional case, the substitution $v = h_x$ leads to the (nonviscous) Burgers equation

$$v_t + 2\lambda\, v v_x = 0.$$



First-principles derivation of the equation

- The Laplacian term can be interpreted as a result of the surface transport of adsorbed particles caused by the standard Brownian diffusion;
- In several experimental situations a hopping mechanism of surface transport is present which necessitates augmentation of the Laplacian by a nonlocal term modeled by a Lévy stochastic process;
- The quadratic nonlinearity is a result of truncation of a series expansion of a more general, physically justified, nonlinear even function.

Fractal Hamilton-Jacobi-KPZ equation

The surface transport may be caused, besides the standard Brownian diffusion, by a hopping mechanism modeled by a Lévy flight.

KPZ= Kardar, Parisi and Zhang (1986) and the standard Brownian diffusion
Hopping mechanism in KPZ introduced by Mann and Woyczyński (2001)

This leads to the nonlinear nonlocal equation

$$u_t = -\mathcal{L}u + \lambda|\nabla u|^q$$

where $\mathcal{L}$ is the Lévy operator and

$$\lambda|\nabla u|^q = \lambda\left(|\partial_{x_1} u|^2 + ... + |\partial_{x_n} u|^2\right)^{q/2}$$

Here, $q = 2$ is the best choice from the physical point of view.

For the intensity constant $\lambda \in \mathbb{R}$, we distinguish two cases:

- *the deposition case* $\lambda > 0$ (the intensity of the ballistic rain),
- *evaporation case* for $\lambda < 0$.

Preliminary result

Theorem

Assume that

$$\mathcal{L} \sim (-\Delta)^{\alpha/2}, \quad \textit{for} \quad \alpha \in (1, 2].$$

For every initial datum

$$u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \quad \textit{and} \quad \lambda \in \mathbb{R}$$

the initial value problem for the fractal Hamilton-Jacobi-KPZ equation

$$u_t = -\mathcal{L}u + \lambda|\nabla u|^q$$

has the unique solution in the space

$$\mathcal{X} = C([0,\infty), W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)).$$

Moreover, this solutions satisfies the estimates

$$\|u(t)\|_\infty \le \|u_0\|_\infty \quad \textit{and} \quad \|\nabla u(t)\|_\infty \le \|\nabla u_0\|_\infty$$

for all $t \ge 0$.

Maximum principle

Lemma

Let $\varphi \in C_b^2(\mathbb{R}^n)$. *Assume that the sequence* $\{x_n\}_{n\ge1} \subset \mathbb{R}^n$ *satisfies*

$$\varphi(x_n) \to \sup_{x\in\mathbb{R}^n} \varphi(x).$$

Then

$$\lim_{n\to\infty} \nabla\varphi(x_n) = 0 \qquad \textit{and} \qquad \limsup_{n\to\infty} -\mathcal{L}\varphi(x_n) \le 0.$$

Proof.
Since $D^2\varphi$ is bounded there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \varphi(x) \ge \varphi(x_n + z) \ge \varphi(x_n) + \nabla\varphi(x_n) - C|z|^2.$$

Since $\nabla\varphi(x_n)$ is bounded, passing to the subsequence, we can assume that

$$\nabla\varphi(x_n) \to p.$$

Hennce, passing to the limit in the inequality above we obtain

$$0 \ge p \cdot z - C|z|^2$$

for every $z \in \mathbb{R}^n$. Chosing $z = tp$ and letting $t \to 0^+$, we have $p = 0$.

Maximum principle

Now, we prove that

$$\limsup_{n\to\infty} -\mathcal{L}\varphi(x_n) \le 0.$$

Note first that

$$\varphi(x_n + z) - \varphi(x_n) \le \sup_{x\in\mathbb{R}^n} \varphi - \varphi(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\limsup_{n\to\infty} \Big(\varphi(x_n + z) - \varphi(x_n)\Big) \le 0$$

and

$$\limsup_{n\to\infty} \Big(\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z\Big) \le 0.$$

Hence, it suffices to use the Fatou lemma in the expression

$$\mathcal{L}\varphi(x_n) = \int_{\mathbb{R}^n} \left(\varphi(x_n - z) - u(x_n) - z \cdot \nabla\varphi(x_n)\mathbb{1}_{\{|z|<1\}}(z)\right) \Pi(dz).$$

□

Maximum principle

Theorem (Droniou & Imbert (2007))
Let

$$\mathcal{L}\varphi(x) = \int_{\mathbb{R}^n} \big(\varphi(x-z) - \varphi(x) - z\cdot\nabla\varphi(x)\mathbb{1}_{\{|z|<1\}}(z)\big)\,\Pi(dz).$$

Assume that

$$u \in C_b(\mathbb{R}^n \times [0,T]) \cap C_b^2(\mathbb{R}^n \times [\varepsilon, T])$$

is the solution of the equation

$$u_t = -\mathcal{L}u + b(x,t)\nabla u,$$

where $b = b(x,t)$ is given and sufficiently regular.
Then

$$u(x,0) \le 0 \qquad \textit{implies} \qquad u(x,t) \le 0.$$

Proof.
The function

$$\Phi(t) = \sup_{x\in\mathbb{R}^n} u(x,t)$$

is well-defined and continuous.
Claim: Φ is locally Lipschitz and $\Phi'(t) \le 0$ almost everywhere.

Lipschitz continuity of Φ

For every $\varepsilon > 0$ there is x_ε such that

$$\sup_{x\in\mathbb{R}^n} u(x,t) = u(x_\varepsilon, t) + \varepsilon.$$

Now, we fix t, s and we suppose that $\Phi(t) \ge \Phi(s)$. Then

$$\begin{aligned} 0 \le \Phi(t) - \Phi(s) &= \sup_x u(x,t) - \sup_x u(x,s) \\ &\le \varepsilon + u(x_\varepsilon, t) - u(x_\varepsilon, s) \\ &\le \varepsilon + \sup_x |u(x,t) - u(x,s)| \\ &\le \varepsilon + |t-s| \sup_{x,t} |\nabla_t u(x,t)|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the function Φ is locally Lipschitz, hence it is **differentiable almost everywhere**.

$\Phi'(t) \le 0$ almost everywhere

Now, we differentiate

$$\Phi(t) = \sup_{x\in\mathbb{R}^n} u(x,t).$$

By the Taylor expansion, for $0 < s < t$, we have

$$u(x,t) \le u(x, t-s) + s\partial_t u(x,t) + Cs^2.$$

Hence,

$$u(x,t) \le \sup_x u(x,t-s) + s\Big(-\mathcal{L}u(x,t) + b(x,t)\nabla u(x,t)\Big) + Cs^2.$$

Substitute $x = x_n$, where $u(x_n, t) \to \sup_x u(x,t)$ as $n \to \infty$.
Passing to the limit, we obtain

$$\sup_x u(x,t) \le \sup_x u(x,t-s) + Cs^2,$$

so

$$\frac{\Phi(t) - \Phi(s)}{s} \le Cs.$$

When $s \to 0$, we conclude $\Phi'(t) \le 0$.

□

Mass evolution

Fractal Hamilton-Jacobi-KPZ equations

$$u_t = -\mathcal{L}u + \lambda|\nabla u|^q$$

"Mass" of the solution

$$\begin{aligned} M(t) &= \|u(t)\|_1 = \int_{\mathbb{R}^N} u(x,t)\,dx \\ &= \int_{\mathbb{R}^N} u_0(x)\,dx + \lambda \int_0^t \int_{\mathbb{R}^N} |\nabla u(x,s)|^q\,dxds \end{aligned}$$

We have

- $M(t) \nearrow$ in the deposition case, i.e., for $\lambda > 0$,
- $M(t) \searrow$ in the evaporation case, i.e., for $\lambda < 0$.

(This is the joint work with W.A. Woyczyński (2008)).

Deposition case: $\lambda > 0$ and the increasing mass

$$M(t) = \int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx + \lambda \int_0^t \int_{\mathbb{R}^n} |\nabla u(x,s)|^q\,dxds$$

Deposition case: $\lambda > 0$

Theorem

Let $\lambda > 0$ and

$$1 < q \le \frac{N+\alpha}{N+1}.$$

Assume that

$$\mathcal{L} \sim (-\Delta)^{\alpha/2} \quad \textit{for} \quad \alpha \in (1,2].$$

If $u = u(x,t)$ is a solution with an initial datum satisfying conditions

$$0 \le u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N),$$

and $u_0 \not\equiv 0$, then

$$\lim_{t\to\infty} M(t) = +\infty.$$

Deposition case: $\lambda > 0$

Theorem

Under the assumptions of the above theorem and if $n \ge 2$ there exists $T_0 = t_0(u_0)$ such that, for all $t \ge t_0(u_0)$

$$M(t) \ge \begin{cases} C(q)\lambda M_0^q t^{(N+\alpha-(N+1)q)/\alpha}, & \textit{for} \quad 1 \le q < \frac{N+\alpha}{N+1}; \\ C(q)\lambda M_0^q \log t, & \textit{for} \quad q = \frac{N+\alpha}{N+1}. \end{cases}$$

Proof.

Since λ and u_0 are nonnegative, it follows that

$$u(t) = e^{-t\mathcal{L}}u_0 + \lambda \int_0^t e^{-(t-\tau)\mathcal{L}} |\nabla u(\tau)|^q\,d\tau \ge e^{-t\mathcal{L}}u_0.$$

Moreover,

$$\lambda^{-1}M(t) = \lambda^{-1}\|u(t)\|_1 \ge \int_0^t \|\nabla u(\tau)\|_q^q\,d\tau.$$

Hence, by the Sobolev inequality, we obtain

$$\lambda^{-1}M(t) \ge C\int_0^t \|u(\tau)\|^q_{Nq/(N-q)}\,d\tau \ge C\int_0^t \|e^{-\tau\mathcal{L}}u_0\|^q_{Nq/(N-q)}\,d\tau.$$

□

Deposition case: $\lambda > 0$

Theorem

Let $\lambda > 0$ and

$$q > \frac{N+\alpha}{N+1}.$$

Assume that

$$\mathcal{L} \sim (-\Delta)^{\alpha/2} \quad \textit{for} \quad \alpha \in (1,2].$$

If

either $\|u_0\|_1$ *or* $\|\nabla u_0\|_\infty$ *is sufficiently small*

then

$$\lim_{t\to\infty} M(t) = M_\infty < \infty.$$

Deposition case: $\lambda > 0$

Idea of the proof.

We work with the integral equation

$$\nabla u(t) = \nabla e^{-t\mathcal{L}} u_0 + \lambda \int_0^t \nabla e^{-(t-\tau)\mathcal{L}} |\nabla u(\tau)|^q \, d\tau$$

in order to show that

$$\|\nabla u(t)\|_q^q \le C(1+t)^{-\kappa}$$

for some $\kappa > 1$, provided

either $\|u_0\|_1$ or $\|\nabla u_0\|_\infty$ is sufficiently small.

Deposition case: $\lambda > 0$

Remark

If

$$\mathcal{L} = (-\Delta)^{\alpha/2},$$

it suffices only to assume that the quantity

$$\|u_0\|_1 \|\nabla u_0\|_\infty^{(q(N+1)-\alpha-N)/(\alpha-1)}$$

is small

Deposition case: $\lambda > 0$.

Theorem

Let $\lambda > 0$ *and*

$$q \ge 2.$$

Suppose that the Lévy diffusion operator $\mathcal{L}$ *has a non-degenerate Brownian part:*

$$\mathcal{L} \sim -\Delta + (-\Delta)^{\alpha/2} \quad \textit{for} \quad \alpha \in (1, 2].$$

Then, **each nonnegative solution** *with an initial datum*

$$u_0 \in W^{1,\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$$

has the mass $M(t) = \int_{\mathbb{R}^N} u(x,t)\, dx$ *increasing to a finite limit*

$$\lim_{t\to\infty} M(t) = M_\infty < \infty.$$

Idea of the proof. A priori estimates and **"classical"** integration by parts.

Deposition case

Remarks

The smallness assumption imposed above seems to be necessary.

- For $\mathcal{L} = -\Delta$, $\lambda > 0$, and

  $$\frac{N+2}{N+1} < q < 2,$$

  there exists a solution such that

  $$\lim_{t\to\infty} M(t) = +\infty$$

  (cf. Ben-Artzi, Souplet & Weissler (2002))

- if $\|u_0\|_1$ and $\|\nabla u_0\|_\infty$ are "large", then the large-time behavior of the solution is dominated by the nonlinear term, hence $M_\infty = \infty$. (Benachour, K. & Laurençot (2004))

Deposition case

Conjectures

- Analogous results hold true at least for $\mathcal{L} = (-\Delta)^{\alpha/2}$ and for q satisfying
$$\frac{N+\alpha}{N+1} < q < \alpha.$$
- The critical exponent $q = 2$ for $\mathcal{L} = -\Delta$ should be replaced by $q = \alpha$. In this case, for $q \geq \alpha$ and as $t \to \infty$, the mass of **every** nonnegative solution converges to a finite limit.

Evaporation case: $\lambda < 0$ and the decreasing mass

$$M(t) = \int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} u_0(x)\, dx + \lambda \int_0^t \int_{\mathbb{R}^n} |\nabla u(x,s)|^q \, dxds$$

Evaporation case: $\lambda < 0$

Theorem
Let $\lambda < 0$ and
$$1 \leq q \leq \frac{N+\alpha}{N+1}.$$
Assume that
$$\mathcal{L} \sim (-\Delta)^{\alpha/2} \quad \text{for} \quad \alpha \in (1,2].$$
If u is a **nonnegative** solution with an initial datum satisfying
$$0 \leq u_0 \in W^{1,\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$$
then
$$\lim_{t\to\infty} M(t) = 0.$$

Evaporation case: $\lambda < 0$

When q is greater than the critical exponent, the diffusion effects prevails for large times.

Theorem
Let $\lambda < 0$ and
$$q > \frac{N+\alpha}{N+1}.$$
Assume that
$$\mathcal{L} \sim (-\Delta)^{\alpha/2} \quad \text{for} \quad \alpha \in (1,2].$$
If u is a **nonnegative** solution with an initial datum satisfying
$$0 \leq u_0 \in W^{1,\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$$
then
$$\lim_{t\to\infty} M(t) = M_\infty > 0.$$

Evaporation case: $\lambda < 0$

Remarks

- The proof of Theorem above is based on the decay estimates of $\|\nabla u(t)\|_p$.
- As was the case for $\lambda > 0$, we can significantly simplify the reasoning for Lévy operators $\mathcal{L}$ with nondegenerate Brownian part and $q \geq 2$.

Selfsimilar asymptotics

When the mass $M(t)$ tends to a finite limit M_∞, as $t \to \infty$, the solutions to Cauchy problem for the Fractal Hamilton-Jacobi-KPZ equation display a self-similar asymptotics dictated by the fundamental solution of the linear equation

$$u_t + (-\Delta)^{\alpha/2} u = 0$$

which given by the formula

$$\begin{aligned} p_\alpha(x,t) &= t^{-N/\alpha} p_\alpha(xt^{-1/\alpha}, 1) \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\xi} e^{-t|\xi|^\alpha} \, d\xi. \end{aligned}$$

Selfsimilar asymptotics

Theorem
Let $u = u(x,t)$ be a solution with $u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, and with the Lévy operator $\mathcal{L}$ satisfying

$$\mathcal{L} \sim (-\Delta)^{\alpha/2} \quad \text{for} \quad \alpha \in (1,2].$$

If $\lim_{t\to\infty} M(t) = M_\infty$ exists and is finite then

$$\lim_{t\to\infty} \|u(t) - M_\infty p_\alpha(t)\|_1 = 0.$$

If, additionally,

$$\|u(t)\|_p \leq Ct^{-N(1-1/p)/\alpha}$$

for some $p \in (1,\infty]$, all $t > 0$, and a constant C then, for every $r \in [1,p)$,

$$\lim_{t\to\infty} t^{N(1-1/r)/\alpha} \|u(t) - M_\infty p_\alpha(t)\|_r = 0.$$

The case $\alpha = 2$, $\mathcal{L} = -\Delta$, and $M_\infty \in \{0, \infty\}$

Deposition case and $M_\infty = +\infty$
The large time asymptotics is decribed by the self-similar solution

$$z(x,t) = \left(K - (q-1)\, q^{-q/(q-1)} \left(\frac{|x|}{t^{1/q}} \right)^{q/(q-1)} \right)^+$$

of the equation

$$z_t = |\nabla z|^q.$$

Evaporation case and $M_\infty = 0$
The large time asymptotics is decribed by the self-similar solution

$$w(x,t) = t^{-a/2} W(xt^{-1/2}) \quad \text{with} \quad a = \frac{2-q}{q-1}$$

of the equation

$$u_t = \Delta u - |\nabla u|^q.$$

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