Gamma-convergence of gradient flows

Courant Institute of Mathematical Sciences,
251 Mercer St, New York NY 10012, USA.
serfaty@cims.nyu.edu
http://www.math.nyu.edu/faculty/serfaty

Gamma-convergence of gradient flows and applications to Ginzburg-Landau vortex dynamics

Sylvia Serfaty

July 20, 2012

Abstract

We present in parallel an abstract method of Γ-convergence of gradient flows, designed to pass to the limit in PDEs which are steepest-descent for functionals which have an asymptotic Γ-limit energy; together with the application to the Ginzburg-Landau energy. We give schematic proofs of the Γ-convergence results for Ginzburg-Landau and of the derivation of the dynamical law of vortices through the abstract method.

I Introduction

I.1 Presentation of the Ginzburg-Landau model

The Ginzburg-Landau energy was introduced by Ginzburg and Landau in the 50s as a model for superconductivity. It was first a phenomenological theory, but it was later derived (in a certain limit) from the microscopic (quantic) theory of Bardeen-Cooper-Schrieffer. It is now a widely accepted model, which has earned its inventors the Physics Nobel Prize (to Ginzburg, Abrikosov, and in 2003 Ginzburg). Another motivation is the modelling of superfluidity (a phenomenon very close to superfluidity, both mathematically and physically, with a joint Nobel Prize for Legett in 2003) and of Bose-Einstein condensates in rotation (Bose-Enstein condensates were predicted by Bose and Einstein in the early 20th century, and only first realized experimentally in the 90's (it was worth another Nobel Prize...). All these physical phenomena have in common the appearance of *topological vortices*, which are the main object of our study.

Superconductors have this striking feature that "they repel an applied magnetic field" (this is called the Meissner effect). This is true at least when the intensity of the applied field h_{ex} is not too large; when it becomes larger than a first critical field H_{c_1}, then the first *vortices* appear and the magnetic field penetrates through them; when the applied field is further raised, there are more and more vortices, until superconductivity is totally destroyed and the magnetic field completely penetrates the sample. For further reference, we refer to the physics literature, e.g. [24, 7].

The samples are 3D, however, we will consider only the 2D model for simplicity (it already contains most of the important features). The 2D Ginzurg-Landau energy in non-dimensional form is

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\text{curl } A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2. \tag{1}$$

Here Ω denotes a smooth bounded and *simply connected* domain corresponding to the cross-section of the sample (assuming everything is translation-invariant in the third direction). The function $u : \Omega \to \mathbb{C}$ is called the *order parameter*, $|u(x)|^2 \le 1$ indicates the local (normalized) density of superconducting electrons (the "Cooper pairs"). Where $|u(x)| \sim 1$ it is the superconducting phase, where $|u(x)| \sim 0$, it is the normal phase. This order parameter is coupled, in a *gauge-invariant* fashion, to a magnetic potential $A : \Omega \to \mathbb{R}^2$, and the function $h = \text{curl}\, A = \partial_2 A_1 - \partial_1 A_2$ is the induced magnetic field in the sample. The real parameter h_{ex} is the intensity of the external applied magnetic field.

The parameter $1/\varepsilon$ is called the Ginzburg-Landau parameter, it is a dimensionless parameter depending on the material (ratio of two characteristic lengthes). When $1/\varepsilon$ is large enough, we are in the category of "type-II" superconductors, when $\varepsilon \to 0$, they are sometimes called "extreme type-II" (or this is also called the "London limit"). This is the asymptotic regime we will be interested in.

I.1.1 Vortices

Vortices are objects centered at zeros of the order parameter u which carry a nonzero topological degree. Typically, around a vortex centered at a point x_0, u "looks like" $u = \rho e^{i\varphi}$ with $\rho(x_0) = 0$ and $\rho = f(\frac{|x-x_0|}{\varepsilon})$ where $f(0) = 0$ and f tends to 1 as $r \to +\infty$, i.e. its characteristic core size is ε, and

$$\frac{1}{2\pi}\int \frac{\partial\varphi}{\partial\tau} = d \in \mathbb{Z}$$

is an integer, called the *degree of the vortex.* For example $\varphi = d\theta$ where θ is the polar angle centered at x_0 yields a vortex of degree d. We have the important relation

$$\text{curl}\,\nabla\varphi = 2\pi \sum_i d_i \delta_{a_i}$$

where the a_i's are the centers of the vortices and the d_i their degrees.

I.1.2 Simplified model (no magnetic coupling)

A simplified model consists in taking $A = 0$ and $h_{\text{ex}} = 0$, then the energy reduces to

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \tag{2}$$

with still $u : \Omega \to \mathbb{C}$. Critical points of this energy are solutions of

$$-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2). \tag{3}$$

The first main study of this functional was done by Bethuel-Brezis-Hélein in the book [2]. Since then, a large literature on it has developed. In these notes, for simplicity, we will focus only on this energy (2), however all our results can be extended to the case of (1).

For more reference on (1), and results on its minimizers, their vortices, critical fields, etc, we refer to the monograph [20] and the references therein. In what follows we will often be a little imprecise in the statements for the sake of simplicity, however exact and rigorous corresponding statements can easily be found in the references.

I.2 Gamma-convergence

Let us now present the totally independent concept of Γ-convergence. It was introduced by DeGiorgi in the 70s, it served to unify various notions of variational convergence.

The idea is dimension-reduction: when there is a small parameter $\varepsilon \to 0$, reduce the minimization of some original functionals E_ε to that of a limiting energy F, defined on a lower-dimensional space.

A celebrated example of Γ-convergence was the case of the energy of the "gradient theory of phase-transitions" studied in the 80's by Modica and Mortola (see also Sternberg):

$$M_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega (1-u^2)^2 \qquad u : \Omega \to \mathbb{R} \tag{4}$$

that is the same as (2) but for *real-valued* functions. It was established that if for a family u_ε, $M_\varepsilon(u_\varepsilon) \le C$, then, up to extraction of a subsequence, $u_\varepsilon \to u_0$ in $BV(\Omega)$ (the space of functions of bounded variation), with u_0 valued in $\{1, -1\}$ and

$$M_\varepsilon \xrightarrow{\Gamma} \frac{8}{3} \text{ per } \gamma = \frac{8}{3} \text{ per } (\partial\{u_0 = 1\}) = \frac{4}{3}\int |Du_0| = \frac{4}{3}\|u_0\|_{BV}$$

where γ (typically a codimension 1 object), is the interface between $\{u_0 = 1\}$ and $\{u_0 = -1\}$.

A trick that was used was to write $a^2 + b^2 \ge 2ab$ (with equality if $a = b$) hence

$$\varepsilon \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int_\Omega (1-u_\varepsilon^2)^2 \ge 2\int |\nabla u_\varepsilon||1-u_\varepsilon^2| \ge 2\int_\Omega \left|\nabla\left(u_\varepsilon - \frac{u_\varepsilon^3}{3}\right)\right|$$

and thus passing to the limit,

$$\varliminf_{\varepsilon\to 0} M_\varepsilon(u_\varepsilon) \ge \frac{8}{3} \text{ per } \gamma.$$

Conversely, given an interface γ (a curve if $\Omega \subset \mathbb{R}^2$), one can construct u_ε such that $M_\varepsilon(u_\varepsilon) \to \frac{8}{3}\mathrm{per}(\gamma)$. This necessitates to paste transversally to γ the optimal profile such that $a = b$ above i.e. $\sqrt{\varepsilon}|\nabla u| = \frac{1}{\sqrt{\varepsilon}}\left|1 - u^2\right|$, that is

$$u_\varepsilon(x_1, x_2) \simeq \tanh\left(\frac{x_1}{\varepsilon}\right)$$

where x_1 is the coordinate in the direction normal to γ.

Definition 1 (Γ-convergence) *A family of functionals E_ε (defined on $\mathcal{M}_\varepsilon$) Γ-converges to a functional F (defined on $\mathcal{N}$) if*

1. *If $E_\varepsilon(u_\varepsilon) \leq C$ then up to extraction of a subsequence, $u_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N}$, and for every $u_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N}$ we have*

$$\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u)$$

2. *For every $u \in \mathcal{N}$, there exists $u_\varepsilon \in \mathcal{M}_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N}$ such that*

$$\overline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \leq F(u)$$

The sense of convergence S is to be specified beforehand. It can be a weak or strong convergence of u_ε, it can also be a convergence of a nonlinear function of u_ε.

In the case of the functional M_ε, one should take $u_\varepsilon \stackrel{S}{\rightharpoonup} \gamma \iff u_\varepsilon \to u_0$ in $L^1(\Omega)$ with $Du_0 = \frac{4}{3}\mathcal{H}^{n-1}\lfloor\gamma$ where γ denotes a codimension one rectifiable current, and $\mathcal{H}$ the Hausdorff measure.

Γ-convergence thus requires two conditions: a lower bound, usually obtained via abstract arguments (together with a compactness result), and an upper bound, usually obtained via explicit constructions.

Proposition 1 *If E_ε Γ-converges to F and u_ε minimizes E_ε with $E_\varepsilon(u_\varepsilon) \leq C$, then, up to extraction $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ and u minimizes F.*

By 1) of the definition, after extraction, $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ and $\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u)$. Let us assume that there exists $u_0 \in \mathcal{N}$ such that $F(u_0) < F(u)$, then by 2) of the definition, there exists v_ε such that $\overline{\lim}_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) \leq F(u_0) < F(u) \leq \underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$. Thus for ε small enough, we find $E_\varepsilon(v_\varepsilon) < E_\varepsilon(u_\varepsilon)$ contradicting the minimality of u_ε. Hence u must minimize F.

In other words "minimizers converge to minimizers". Minimizing M_ε defined over $H^1(\Omega, \mathbb{R})$ for example reduces to minimizing $F(\gamma) = per(\gamma)$ defined over finite-perimeter sets. It thus achieves a dimension-reduction (since the set of finite-perimeter objects has, somewhat, a lower dimension than $H^1(\Omega, \mathbb{R})$). In general, not much more can be said. For example u_ε local minimizer of E_ε *does not imply* $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ local minimizer; or u_ε critical point of E_ε *does not imply* $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ critical point of F. It is easy to construct finite-dimensional counter-examples.

I.3 Γ-convergence of Ginzburg-Landau

I.3.1 Energy lower bound

To obtain a lower bound (and thus a Γ-convergence result) for the Ginzburg-Landau functional (2) is more difficult than for M_ε (the $a^2 + b^2 \geq 2ab$ trick doesn't work).

Let us present (formally) some essential ingredients of the analysis of [2]. What is the cost of a radial vortex of degree d of the form $f\left(\frac{r}{\varepsilon}\right)\ e^{i\,d\theta}$? First, formally

$$\frac{1}{2}\int_{B_R}|\nabla u|^2 \geq \frac{1}{2}\int_{R\geq|x|\geq\varepsilon}|f|^2\,\frac{d^2}{r^2}\,r\,dr\,d\theta = \pi d^2\int_\varepsilon^R\frac{dr}{r} = \pi d^2\log\frac{R}{\varepsilon} \tag{5}$$

where we have assumed that f is close to 1 for $|x| \geq \varepsilon$. In fact this bound is optimal, at least in the case $d = \pm 1$ as can be seen: if $u \in \mathbb{S}^1$, $u = e^{i\varphi}$, and $|\nabla u| = |\nabla\varphi|$, so

$$\begin{aligned}\frac{1}{2}\int_{R\geq|x|\geq\varepsilon}|\nabla u|^2 &\geq \frac{1}{2}\int_\varepsilon^R\left(\int_{\partial B_r}\left|\frac{\partial\varphi}{\partial\tau}\right|^2\right)dr\\ &\geq \frac{1}{2}\int_\varepsilon^R\left(\left(\int_{\partial\,B_r}\frac{\partial\varphi}{\partial\tau}\right)^2\frac{1}{2\pi r}\right)dr \quad \text{(by Cauchy-Schwarz)}\\ &\geq \frac{1}{2}\,\frac{4\pi^2}{2\pi}\int_\varepsilon^R\frac{dr}{r} = \pi\,\log\frac{R}{\varepsilon}\end{aligned}$$

valid for any degree ±1 vortex (not necessarily radial). Vortices of degree > 1 cost more energy than several vortices of degree 1 and are in fact unstable. The cost of $f\left(\frac{r}{\varepsilon}\right)$ imposes the lengthscale ε, and costs only $O(1)$, which is negligible compared to $\log\frac{1}{\varepsilon}$.

If u_ε has vortices at points $a_1^\varepsilon, \cdots, a_n^\varepsilon$, of degrees $d_1, \cdots, d_n$, one expects that

$$E_\varepsilon(u_\varepsilon) \geq \pi\left(\sum_i |d_i|\right)\log\frac{1}{\varepsilon}.$$

In fact, this estimate has been made rigorous under certain conditions in [2], and more generally with the "ball construction method" of Sandier/Jerrard (see [17, 11, 20]).

How to trace the vortices? The easiest way is to use the current $\langle iu, \nabla u\rangle$ (or the "superconducting current" $\langle iu, \nabla_A u\rangle$ for the case with magnetic field) where $\langle .,.\rangle$ denotes the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$, i.e. $\langle iu, \nabla u\rangle = (u \times \partial_1 u, u \times \partial_2 u)$ with $\times$ the vector product in $\mathbb{R}^2$. Writing $u = \rho e^{i\varphi}$ we have (at least formally)

$$\langle iu, \nabla u\rangle = \rho^2\nabla\phi$$

and since ρ is close to 1 on lengthscales ε, the quantity

$$\mathrm{curl}\,\langle iu, \nabla u\rangle = \mathrm{curl}\,(\rho^2\nabla\varphi) \simeq \mathrm{curl}\,\nabla\varphi = 2\pi \sum_i d_i \delta_{a_i} \tag{6}$$

can be used to trace the vortices.

This is also called the Jacobian determinant if written (with differential forms) $Ju = \frac{1}{2} d\langle iu, du\rangle = \frac{1}{2}\langle idu, du\rangle = u_{x_1} \times u_{x_2}$. The approximation is justified as a limit as $\varepsilon \to 0$:

Theorem 1 (see [13, 20]) *Assume $E_\varepsilon(u) \le C|\log\varepsilon|$, then there exists a family of disjoint closed balls $B_i = B(a_i, r_i)$ with $|\log \varepsilon|^{-2} \le \sum r_i \le o(1)$ as $\varepsilon \to 0$, such that*

$$\left\{|u| \le \frac{1}{2}\right\} \subset \cup_i B_i \quad \textit{(the } B_i\textit{'s cover the zeroes of } u_\varepsilon\textit{)}$$

$$\frac{1}{2}\int_{\cup_i B_i} |\nabla u_\varepsilon|^2 \ge \pi \sum_i |d_i| \left(\log \frac{\sum_i r_i}{\varepsilon \sum_i |d_i|} - C\right) \qquad d_i = deg\,(u, \partial B_i) \tag{7}$$

$$\|\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle - 2\pi \sum_i d_i\,\delta_{a_i}\|_{(C_0^{0,\gamma}(\Omega))^*} \le o(1) \quad \textit{as} \quad \varepsilon \to 0 \tag{8}$$

Combining the upper bound $E_\varepsilon(u) \le C|\log\,\varepsilon|$ and the lower bound (7), we deduce that $\sum_i |d_i| \le C$ for a constant C independent of ε and thus the number of vortices of nonzero degree remains bounded independently of ε. Thus, if u_ε is a family of such configurations, once the a_i^ε's are found, we may extract a subsequence such that $\sum_i d_i \delta_{a_i^\varepsilon} \to \sum_i d_i \delta_{a_i}$ in the weak sense of measures. These fixed points $a_1, \cdots, a_n$ are the limiting vortices. We will sometimes write $u = (a_i, d_i)$ for the limiting points+degrees configurations in $(\Omega \times \mathbb{Z})^n$.

Then the Γ-convergence result can simply be written

Theorem 2 *1) Assume u_ε is such that $\frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|} \le C$, then up to extraction*

$$u_\varepsilon \overset{s}{\rightharpoonup} u = (a_i, d_i) \quad \textit{in the sense} \quad \mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i\,\delta_{a_i} = 2J$$

and

$$\varliminf_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|} \ge \pi \sum_{i=1}^n |d_i| = \|J\|$$

2) Conversely given any $(a_i, d_i) \in (\Omega\times\mathbb{Z})^n$, there exists u_ε such that $\varlimsup_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|} \le \pi \sum_{i=1}^n |d_i| = \|J\|$.

This result is not very interesting since it reduces minimizing E_ε to minimizing the number of points!... It is mostly interesting in higher dimensions. Then, in 3D for example, vortices are not points but vortex-lines, and the Jacobian $Ju_\varepsilon =$

$\frac{1}{2}d(iu_\varepsilon, du_\varepsilon)$ can be seen as a current carried by the vortex-line, converging to a π times integer-multiplicity dimension 1 rectifiable current (i.e. line) J and

$$\frac{E_\varepsilon}{|\log\,\varepsilon|} \xrightarrow{\Gamma} \|J\| = \text{length of line (or surface...)}$$

is the lower bound of Γ-convergence (see [13] and Section **??**). Thus, Γ-convergence reduces to minimizing the length of the line, leading to straight lines, a nontrivial problem. In higher dimensions, it leads to codimension 2 minimal currents similarly to M_ε (see [16, 3]).

In fact, in order for the problem to become interesting in 2D, we need to impose some boundary conditions, for example $u_\varepsilon = g\ \partial\Omega$ with $\deg g \neq 0$ so that there *have* to be vortices, and to look at the next order of the energy in the expansion. This rather arbitrary boundary requirement is in contrast with the case of the full functional (1), for which the natural boundary condition is Neumann, and vortices appear due to the applied magnetic field.

I.3.2 Renormalized energy

Let us return to lower bounds in order to look for the next order term in the energy (still with formal arguments). Cutting out holes $\cup_i B(a_i,\rho)$ of fixed size ρ around the limiting vortices a_i, we may assume that $|u| \sim 1$ in $\Omega\backslash \cup_i B(a_i,\rho) = \Omega_\rho$, and that $u = e^{i\varphi}$, with φ a real-valued function, *not* single-valued though (i.e. only defined modulo 2π). Minimizing the energy outside of the holes amounts to solving

$$\min_{\substack{u:\Omega_\rho\to\mathbb{S}^1\\ u=g\ \text{on}\ \partial\Omega\\ deg(u,\partial B(a_i,\rho))=d_i}} \frac{1}{2}\int_{\Omega_\rho} |\nabla u|^2.$$

This is a harmonic map problem, whose solution is given in terms of φ by

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega_\rho \\ \frac{\partial\varphi}{\partial\tau} & \text{given on } \partial\Omega \\ \int_{\partial B(a_i,\rho)} \frac{\partial\varphi}{\partial\tau} = 2\pi d_i. \end{cases}$$

and in terms of the harmonic conjuguate Φ such that $\nabla\varphi = \nabla^\perp\Phi$, by

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega_\rho \\ \dfrac{\partial\Phi}{\partial n} & \text{given on } \ \partial\Omega \ \text{ or } \ \phi = 0 \text{ on } \partial\Omega \text{ for Neumann b.c.} \\ \displaystyle\int_{\partial B(a_i,\rho)} \frac{\partial\Phi}{\partial n} = 2\pi d_i. \end{cases}$$

As $\rho \to 0$, Φ behaves like the solution of

$$\begin{cases} \Delta\Phi_0 = 2\pi\sum_i d_i\delta_{a_i} \quad \Omega \\ \dfrac{\partial\Phi_0}{\partial n} \text{ given } \ \text{ or } \ \Phi_0 = 0 \text{ on } \partial\Omega \text{ for Neumann b.c.} \end{cases}$$

Introducing the Green's kernel associated to Ω (with the right boundary condition), which has a $\log|x-y|$ type singularity, and its regular part $S(x,y) = 2\pi G(x,y) - \log|x-y|$, we have

$$\Phi_0(x) = 2\pi \sum_j d_j G(x, a_j).$$

With this relation, $\int_\Omega |\nabla \Phi_0|^2$ is infinite but would write formally like

$$\int_\Omega |\nabla \Phi_0|^2 = -2\pi \sum_i d_i \Phi_0(a_i) = -4\pi^2 \sum_{i,j} d_i d_j G(a_i, a_j).$$

Now we wish to estimate $\frac{1}{2}\int_{\Omega_\rho} |\nabla \varphi|^2 = \frac{1}{2}\int_{\Omega_\rho} |\nabla \Phi|^2$ and it is approximately equal to

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla \Phi_0|^2 \simeq \pi \sum_i d_i^2 \log \frac{1}{\rho} + W_{\mathbf{d}}(a_1, \cdots, a_n) + o(1) \quad \text{as } \rho \to 0 \tag{9}$$

where

$$W_{\mathbf{d}}(a_1, \cdots, a_n) = -\pi \sum_{i \neq j} d_i d_j \log|a_i - a_j| - \pi \sum_{i,j} S(a_i, a_j). \tag{10}$$

The function W was called the *renormalized energy* in [2]. It contains the (logarithmic) interaction energy between the vortices: we see that vortices with degrees of same sign repel each other while vortices with degrees of opposite signs attract. The $d^2 \log \frac{1}{\rho}$ term corresponds to the self-interaction, or cost of the vortex of core of size ρ, it is what replaces the infinite term in the formal calculation.

Now (9) is a good estimate for the optimal energy outside of the holes, while the energy in holes of size ρ was estimated through (5). Combining these estimates, we are led to the major result of [2]:

Theorem 3 ([2]) *Assume $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$ and $u_\varepsilon = g$ on $\partial\Omega$, with deg $g \neq 0$. Then, up to extraction,*

$$\operatorname{curl} \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i} \quad d_i \in \mathbb{Z}$$

and

$$E_\varepsilon(u_\varepsilon) \ge \pi \sum_{i=1}^n |d_i| \log \frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1, \ldots, a_n) + o(1) \quad \textit{as} \quad \varepsilon \to 0$$

So for a given degree $d > 0$ on the boundary, in order to minimize the energy, one needs to choose d vortices of degree $+1$, and then to minimize the remaining interaction term W which is independent of ε and governs the locations of the limiting vortices.

From now on, we will reduce to the case $d_i = \pm 1$ and will use

Theorem 4 *1. Assume $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$ and $u_\varepsilon = g$ on $\partial\Omega$ or $\frac{\partial u_\varepsilon}{\partial n} = 0$ on $\partial\Omega$, then, up to extraction,*

$$\operatorname{curl} \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i}$$

and if $\forall i$, $d_i = \pm 1$,

$$\varliminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) - \pi n|\log \varepsilon| \ge W_{\mathbf{d}}(a_1, \cdots, a_n).$$

2. For all (a_i, d_i), $d_i = \pm 1$, there exists u_ε such that

$$\varlimsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) - \pi n|\log \varepsilon| \le W_{\mathbf{d}}(a_1, \cdots, a_n).$$

Phrased this way, it is a result of Γ-convergence of $E_\varepsilon - \pi n|\log \varepsilon|$, and the Γ-limit, $W_{\mathbf{d}}$ is *nontrivial*. We thus reduce minimizing E_ε to minimizing $W_{\mathbf{d}}$ which is a *finite-dimensional* problem (interaction of point charges). Thus we see why it is interesting to study this asymptotic limit $\varepsilon \to 0$ because the vortices become point-like and the problem reduces to a finite-dimensional one.

II The abstract result for Γ-convergence of gradient-flows

II.1 The abstract situation

Let E_ε be again a family of functionals defined on $\mathcal{M}_\varepsilon$ (see [19] for an idea of what kind of space $\mathcal{M}_\varepsilon$ should be...) and F be a functional defined on $\mathcal{N}$ such that E_ε Γ-converges to F for the sense of convergence S (in the sense of Definition 1). If we consider a solution of the gradient-flow (or steepest descent) of E_ε on $\mathcal{M}_\varepsilon$ i.e.

$$\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon),$$

does $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ for some $u(t)$ and more importantly, does $u(t)$ satisfy $\partial_t u = -\nabla F(u)$? An example with a positive answer is that of the functional M_ε whose L^2 gradient-flow is the Allen-Cahn equation

$$\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \tag{11}$$

We saw that $M_\varepsilon \overset{\Gamma}{\to} F = \frac{8}{3}\text{per}\,(\gamma)$. In fact it is true that solutions of the Allen-Cahn equation converge to interfaces which evolve according to the gradient flow of that perimeter functional F, which is *mean-curvature flow*. This result is a delicate one, which has been proved with PDE methods (see [8, 6, 9, 10]).

Let us point out that the answer is in general *negative* without further assumptions. Indeed, a necessary condition is that critical points of E_ε should converge to critical points of F, but we already mentioned that when E_ε Γ-converges to F, this is not necessarily true.

We are searching for
- an abstract result
- an energy-based method
- new extra conditions for convergence to occur.

Observe that these have to involve the C^1 (or tangent) structure of the energy landscape, i.e. be conditions on the derivatives of the energy and not only of the energies themselves (otherwise it is easy to perturb the energy by a small perturbation in C^0 which adds new critical points which do not converge to critical points).

We have been sloppy until now, by writing $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$ and calling this the gradient-flow of E_ε. Since we are in infinite dimensions (in general), we need to specify what we mean by gradient, i.e. gradient with respect to which structure. There are many possible choices, each leading to a different gradient-flow. For example, the Allen-Cahn equation (11) above is the gradient flow for M_ε for the L^2 structure. We could consider other structures, such as the gradient-flow with respect to the H^{-1} structure, it is then a totally different dynamics, called the Cahn-Hilliard equation. So, when looking for a result of convergence, we need to specify what the structure for the limiting flow should be (recall that the limiting flow is not taken in the same space, $u_\varepsilon \in \mathcal{M}_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N} \neq \mathcal{M}_\varepsilon$.) Another element that should come into play is possible time-rescalings as we pass to the limit $\varepsilon \to 0$.

II.2 The result

For simplicity we will reduce to the following case: E_ε is family of C^1 functionals defined over $\mathcal{M}$, an open subset of a Banach space $\mathcal{B}$ continuously embedded into a Hilbert space X_ε (or of an affine space associated to a Banach). We assume $E_\varepsilon \xrightarrow{\Gamma} F$, with F a C^1 functional defined over $\mathcal{N}$, open set of a finite-dimensional vector space $\mathcal{B}'$ embedded into a finite-dimensional Hilbert space Y.

Definition 2 *E_ε Γ-converges along the trajectory $u_\varepsilon(t)$ ($t \in [0,T)$) in the sense S to F if there exists $u(t) \in \mathcal{N}$ and a subsequence (still denoted u_ε) such that $\forall t \in [0,T)$, $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ and*

$$\forall t \in [0,T) \quad \varliminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t)).$$

Definition 3 *If $dE_\varepsilon(u)$, differential of E_ε at u, is linear continuous on X_ε, it is uniquely represented by a vector in X_ε, denote it by $\nabla_{X_\varepsilon} E_\varepsilon(u)$ (gradient for the structure X_ε), characterized by*

$$\forall \phi \in X_\varepsilon \quad \frac{d}{dt}_{|t=0} E_\varepsilon(u + t\phi) = dE_\varepsilon(u) \cdot \phi = \langle \nabla_{X_\varepsilon} E_\varepsilon(u), \phi \rangle_{X_\varepsilon}.$$

If this gradient does not exist, we use the convention $\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon} = +\infty$

For example, for the dynamics of Ginzburg-Landau, we wish to study the equation

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2). \tag{12}$$

Let us see how to fit into the previous framework. E_ε is defined on $H^1(\Omega,\mathbb{C})$, we take $\mathcal{B} = H^1(\Omega,\mathbb{C}) \subset L^2(\Omega)$ and we define the X_ε structure by

$$\|\cdot\|_{X_\varepsilon}^2 = \frac{1}{|\log \varepsilon|}\int_\Omega |\cdot|^2 = \frac{1}{|\log \varepsilon|}\|\cdot\|_{L^2(\Omega)}^2,$$

i.e. a rescaled version of L^2. Then $\mathcal{B}$ embeds continuously into X_ε, and the gradient for the structure X_ε is

$$\nabla_{X_\varepsilon} E_\varepsilon(u) = -|\log \varepsilon| \left(\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right).$$

Indeed

$$\begin{aligned} dE_\varepsilon(u)\cdot\phi &= \int_\Omega \phi\left(-\Delta u - \frac{u}{\varepsilon^2}(1-|u|^2)\right) \\ &= \frac{1}{|\log \varepsilon|}\int_\Omega \phi\left(-|\log\varepsilon|\left(\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right)\right) = \langle \phi, \nabla_{X_\varepsilon} E_\varepsilon(u)\rangle_{X_\varepsilon}. \end{aligned}$$

Then the PDE (12) is indeed exactly $\partial_t u = -\nabla_{X_\varepsilon} E_\varepsilon(u)$ i.e. the gradient-flow for the structure X_ε.

Recall that if a solution of

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \tag{13}$$

is smooth enough, we have

$$\begin{aligned} \langle \partial_t u_\varepsilon, \partial_t u_\varepsilon\rangle_{X_\varepsilon} &= -\langle \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t u_\varepsilon\rangle_{X_\varepsilon} \\ &= -\partial_t E_\varepsilon(u_\varepsilon(t)) \\ \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2\,dt &= E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T)). \end{aligned}$$

Definition 4 *A solution of the gradient-flow for E_ε with respect to the structure X_ε on $[0,T)$ is a map $u_\varepsilon \in H^1([0,T), X_\varepsilon)$ such that*

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \in X_\varepsilon \quad \text{for a.e. } t \in [0,T).$$

Such a solution is conservative if $\forall t \in [0,T)$

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2\,ds$$

(this is true if u_ε is smooth enough). If u_ε is such a family of solutions on $[0,T)$ and E_ε Γ-converges to F along $u_\varepsilon(t)$ (in the sense of Definition 2), we define the energy-excess $D(t)$ by $D_\varepsilon(t) = E_\varepsilon(u_\varepsilon(t)) - F(u(t)) \geq o(1)$ and

$$D(t) = \overline{\lim_{\varepsilon\to 0}} D_\varepsilon(t) \geq 0.$$

A family of solutions of the gradient flow is said to be well-prepared initially if $D(0) = 0$.

Recall that F is always a lower bound for E_ε. Also, it is always possible to have well-prepared initial data from assertion 2) (the construction part) of the Γ-convergence definition, Definition 1.

We define similarly the gradient-flow for F for the structure Y. We can now state the abstract result.

Theorem 5 ([19]) *Let E_ε and F be C^1 functionals over $\mathcal{M}$ and $\mathcal{N}$ respectively, $E_\varepsilon \xrightarrow{\Gamma} F$, and let u_ε be a family of conservative solutions of the flow of E_ε*

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \quad on \quad [0,T) \tag{14}$$

with $u_\varepsilon(0) \stackrel{S}{\rightharpoonup} u_0$, along which E_ε Γ-converges to F in the sense of Definition 2.

Assume moreover that 1) and either 2) or 2') below are satisfied:

1) *(lower bound) For a subsequence such that $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$, we have $u \in H^1((0,T),Y)$ and $\forall s \in [0,T)$*

$$\varliminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}(t)\, dt \geq \int_0^s \|\partial_t u\|^2_Y(t)\, dt. \tag{15}$$

2) *For any $t \in [0,T)$*

$$\varliminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(t)\|^2_{X_\varepsilon} \geq \|\nabla_Y F(u(t))\|^2_Y. \tag{16}$$

2') *(construction) If $u_\varepsilon \stackrel{S}{\rightharpoonup} u$, for any $V \in Y$, any v defined in a neighborhood of 0 satisfying*

$$\begin{cases} v(0) &= u \\ \partial_t v(0) &= V \end{cases}$$

there exists $v_\varepsilon(t)$ such that $v_\varepsilon(0) = u_\varepsilon$

$$\varlimsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(0)\|^2_{X_\varepsilon} \leq \|\partial_t v(0)\|^2_Y = \|V\|^2_Y$$

$$\varliminf_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \geq -\frac{d}{dt}_{|t=0} F(v) = -\langle \nabla_Y F(u), V\rangle_Y$$

Then if $D(0) = 0$ (i.e. the solution is well-prepared initially) we have $D(t) = 0$ $\forall t \in [0,T)$, all inequalities above are equalities and $\forall t \in [0,T)$, $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ where

$$\begin{cases} \partial_t u = -\nabla_Y F(u) \\ u(0) = u_0 \end{cases}$$

i.e. u is a solution of the gradient flow for F for the structure Y.

II.3 Interpretation

This theorem means that under conditions 1) and 2), or 1) and 2') (since 2') implies 2)), solutions of the gradient flow of E_ε for the structure X_ε converge to solutions of limiting gradient-flow (for the structure Y) if well-prepared. Let us make a few additional comments:

1. The limiting structure Y is somehow embedded in the conditions 1) and 2). The time rescalings are embedded in X_ε.

2. In general we expect 1) and 2) to be satisfied for any $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ or $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ not necessarily solutions (here we required it only for solutions)

3. 1) and 2) do provide the extra C^1 order conditions on Γ-convergence. 2) in particular implies that critical points converge to critical points.

4. The difficulty is not in proving this theorem but in proving that in specific cases the conditions hold.

II.4 Idea of the proof

Let us see how 1) and 2) imply the result. We assume

$$\begin{cases} \partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \\ \underline{\lim}_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}\, dt \geq \int_0^s \|\partial_t u\|^2_Y \\ \underline{\lim}_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \geq \|\nabla_Y F(u)\|^2_Y \\ u_\varepsilon(0) \stackrel{S}{\rightharpoonup} u_0, \quad \lim E_\varepsilon(u_\varepsilon(0)) = F(u_0) \end{cases}$$

Then, for all $t < T$ we may write

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) &= -\int_0^t \langle \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(s)), \partial_t u_\varepsilon(s)\rangle_{X_\varepsilon}\, ds \\ &= \frac{1}{2}\int_0^t \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} + \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}\, ds \\ &\geq \frac{1}{2}\int_0^t \|\nabla_Y F(u)\|^2_Y | \|\partial_t u\|^2_Y\, ds - o(1) \\ &\geq \int_0^t -\langle \nabla_Y F(u(s)), \partial_t u(s)\rangle_Y\, ds - o(1) \qquad (17) \\ &= F(u(0)) - F(u(t)) - o(1) \end{aligned}$$

hence

$$F(u(0)) - F(u(t)) \leq E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) + o(1)$$

But by well-preparedness $E_\varepsilon(u_\varepsilon(0)) = F(u(0)) + o(1)$ thus

$$E_\varepsilon(u_\varepsilon(t)) \leq F(u(t)) + o(1).$$

But, $E_\varepsilon \xrightarrow{\Gamma} F$ implies $\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t))$ therefore we must have equality everywhere and in particular equality in (17), that is

$$\frac{1}{2}\int_0^t \|\nabla_Y F(u)\|_Y^2 + \|\partial_t u\|_Y^2 \, ds = \int_0^t \langle -\nabla_Y F(u(s)), \partial_t u(s)\rangle_Y$$

or

$$\int_0^t \|\nabla_Y F(u) + \partial_t u\|_Y^2 \, ds = 0.$$

Hence, we conclude that $\partial_t u = -\nabla_Y F(u)$, $\forall t$.

The idea is thus to show that the energy decreases at least of the amount expected (i.e. the amount of decrease of F), on the other hand it cannot decrease more because of the Γ-convergence, hence it decreases exactly of the amount expected, all along the trajectory.

Proof of 2') ⟹ 2) (2') is a constructive proof of 2)). Observe that here u_ε does not depend on time.

For every $V \in Y$ we may pick $v(t)$ such that

$$\begin{cases} v(0) & = u \\ \partial_t v(0) & = V \end{cases}$$

i.e. pick a tangent curve to V at u. We assume there exists (we can construct) $v_\varepsilon(t)$ such that

$$\begin{cases} v_\varepsilon(0) = u_\varepsilon \\ \overline{\lim}_{\varepsilon\to 0} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon}^2 \leq \|V\|_Y^2 \\ \underline{\lim}_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \geq -\frac{d}{dt}_{|t=0} F(v) = -\langle \nabla_Y F(u), V\rangle_Y \end{cases}$$

that is a curve $v_\varepsilon(t)$ along which the energy decreases of at least the desired amount. Then, choosing $V = -\nabla_Y F(u)$, we have

$$\langle -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t v_\varepsilon\rangle_{X_\varepsilon} = -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \geq -\langle \nabla_Y F(u), V\rangle_Y = \|\nabla_Y F(u)\|_Y^2$$

thus

$$\begin{aligned} \|\nabla_Y F(u)\|_Y^2 &\leq \langle -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t v_\varepsilon(0))\rangle_{X_\varepsilon} \\ &\leq \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon} \\ &\leq \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} (\|V\|_Y + o(1)) \end{aligned}$$

Recalling that $V = -\nabla_Y F(u)$, we conclude that

$$\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \geq \|\nabla_Y F(u)\|_Y + o(1).$$

The idea was to rely on the fact that steepest descent is characterized as the evolution which maximizes the energy-decrease for a given $\|\partial_t u_\varepsilon\|^2$. We compare it to a test-evolution obtained by "pushing" u_ε in the direction V (and in fact choose the steepest descent direction $V = -\nabla F(u)$), i.e. find a curve $v(t)$ and "lift it" to a curve v_ε that pushes u_ε in direction $-\nabla_Y F(u)$ with a decrease of energy of at least the expected one, and a cost $\|\partial_t v_\varepsilon\|^2$ which is at most the expected one. We can in fact achieve this in such a way that $\partial_t u_\varepsilon(0)$ depends linearly on V. In "pedantic" terms, we show that there exists a linear embedding

$$\begin{aligned} \mathcal{I}_\varepsilon : T_u\mathcal{N} &\longrightarrow T_{u_\varepsilon}\mathcal{M} \\ V &\mapsto \partial_t v_\varepsilon(0) \end{aligned}$$

which is an "almost-isometry" in the sense :

$$\lim_{\varepsilon\to 0} \|\mathcal{I}_\varepsilon(V)\|_{X_\varepsilon} = \|V\|_Y \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon^* \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) = \nabla_Y F(u).$$

II.5 Application to Ginzburg-Landau

In order to retrieve the dynamical law for vortices, we need to prove that conditions 1) and 2') of Theorem 5 can be proved for Ginzburg-Landau. As seen in Theorem 4, we need to consider the energies

$$F_\varepsilon(u) = E_\varepsilon(u) - \pi n|\log\,\varepsilon| = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 - \pi n|\log\varepsilon| \qquad (18)$$

and $F = W$ (the renormalized energy) so that $F_\varepsilon \xrightarrow{\Gamma} F$. The structures we need are

$$\|\cdot\|^2_{X_\varepsilon} = \frac{1}{|\log\varepsilon|}\|\cdot\|^2_{L^2(\Omega)}$$

$$\mathcal{N} = \Omega^n \backslash \text{diagonals} \qquad (19)$$

$$\|\cdot\|^2_Y = \frac{1}{\pi}\|\cdot\|^2_{(\mathbb{R}^2)^n} \qquad (20)$$

and a prescribed number of vortices of a priori fixed degrees ± 1. Applying Theorem 5, we retrieve the dynamical law (first established by Lin and Jerrard-Soner) that the vortices flow according to a rescaled gradient-flow of the renormalized energy:

Theorem 6 ([15, 12, 19]) *Let u_ε be a family of solutions of*

$$\frac{\partial_t u}{|\log\,\varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)$$

with either

$$\left| \begin{array}{ll} u_\varepsilon = g & \text{on } \partial\Omega \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right.$$

such that

$$\text{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle(0) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i^0} \qquad \text{as } \varepsilon\to 0$$

with a_i^0 distinct points in Ω, $d_i = \pm 1$, and

$$E_\varepsilon(u_\varepsilon)(0) - \pi n|\log \varepsilon| \leq W_{\mathbf{d}}(a_i^0) + o(1). \tag{21}$$

Then there exists $T^* > 0$ such that $\forall t \in [0, T^*)$,

$$\mathrm{curl}\,\langle iu, \nabla u\rangle(t) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i(t)}$$

as $\varepsilon \to 0$, with

$$\begin{cases} \dfrac{da_i}{dt} &= -\dfrac{1}{\pi}\nabla_i W_{\mathbf{d}}(a_1(t), \cdots, a_n(t)) \\ a_i(0) &= a_i^0 \end{cases} \tag{22}$$

where T^* is the minimum of the collision time and exit time (from Ω) of the vortices under this law. Moreover $D(t) = 0$ for every $t < T^*$.

Thus, as expected, vortices move along the gradient flow for their interaction W, and this reduces the PDE to a finite dimensional evolution (a system of ODE's). Thus result was obtained in [15, 12] (also in [23] for a certain regime with magnetic field), but with PDE methods, it is reproven in [19] with the Γ-convergence energetic method exposed here.

II.6 Remarks

1. The result holds as long as the number of vortices remains the initial one (so that the limiting configuration $u = (a_1, \cdots, a_n)$ belongs to the same space $\mathcal{N}$). It ceases to apply when there are vortex-collisions or some vortex exits the domain under the law (22), even though these can happen. Then a further analysis is required, see Section IV.1 below.

2. Under the same hypotheses, if u_ε is a solution of the time-rescaled gradient flow $\partial_t u_\varepsilon = -\lambda_\varepsilon \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)$ with $D(0) = 0$ then if $\lambda_\varepsilon \ll 1$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u_0, \forall t$ i.e. there is no motion; while if $\lambda_\varepsilon \gg 1$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u, \forall t$ with $\nabla_Y F(u) = 0$ i.e. there is instantaneous motion to a critical point. Thus, we see that the structure X_ε and the relation 1) in Theorem 5 contain the right time-rescaling to see finite-time motion in the limit. For Ginzburg-Landau without magnetic field, it is necessary to accelerate the time by a $|\log \varepsilon|$ factor in order to see motion of the vortices (this is due to the fact that the renormalized energy W which drives the motion is a lower order term in the energy).

3. We can weaken conditions 1) and 2) to

$$\underline{\lim}_{\varepsilon\to 0} \int_0^t \|\partial_t u_\varepsilon\|^2 \geq \int_0^t \|\partial_t u\|^2 - O(D(t))$$
$$\underline{\lim}_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \geq \|\nabla F(u)\|_Y^2 - O(D(t))$$

where $D(t)$ is the energy-excess, and handle the terms in $D(t)$ in the proof via a Gronwall's lemma (finally obtaining that $D(t) \equiv 0$ if $D(0) = 0$).

4. The method should and can be extended to infinite-dimensional limiting spaces and to the case where the Hilbert structures X_ε and Y (in particular Y) depend on the point, such as $Y_u = L^2_u$, forming a sort of Hilbert manifold structure. It is thus interesting to see how, through Γ-convergence, the structures underlying the gradient-flows can become "curved" at the limit, even though they are not curved originally at the ε level, and also become possibly nonsmooth and nondifferentiable. In fact we can write down an analogue abstract result using the theory of "minimizing movements" of De Giorgi formalized by Ambrosio-Gigli-Savarè [1], a notion of gradient flows on structures which are not differentiable but simply metric structures.

5. The method works for Ginzburg-Landau with or without magnetic field as long as the number of vortices remains bounded. It is more difficult to apply it to other models such as Allen-Cahn, or 3D Ginzburg-Landau, because what misses is a more precise result and understanding on the profile of the defect during the dynamics. For example, for Allen-Cahn, we need to know that the energy-density remains proportional to the length of the underlying limiting curve during the dynamics (which is true a posteriori). It is also an open problem to apply it when the number of vortices is unbounded as $\varepsilon \to 0$.

III Proof of the additional conditions for Ginzburg-Landau

III.1 A product-estimate for Ginzburg-Landau

The relation 1) which relates the velocity of underlying vortices to $\partial_t u_\varepsilon$ can be read

$$\underline{\lim}_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{[0,t]\times\Omega} |\partial_t u_\varepsilon|^2 \, ds \geq \pi \sum_i \int_0^t |d_t \, a_i|^2 \, ds \tag{23}$$

assuming $\operatorname{curl} \langle iu_\varepsilon, \nabla u_\varepsilon \rangle(t) \rightharpoonup 2\pi \sum_i d_i \delta_{a_i(t)}$, as $\varepsilon \to 0$, $\forall t$. This turns out to hold as a general relation, without asking the configurations to solve any particular equation. It is related to the topological nature of the vortices.

It can be embedded into the more general class of results of lower-bounds for Ginzburg-Landau functionals. The setting is now Ω a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) (we will need $n = 3$) and still $E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2$. We define the "current" ju associated to u as the 1-form $ju = \langle iu, du \rangle = \sum_k \langle iu, \partial_k \, u \rangle dx_k$. Then the Jacobian Ju is the 2-form

$$Ju = \frac{1}{2} d(ju) = \frac{1}{2} d\langle iu, du \rangle$$

It can be identified to an $(n-2)$-dimensional current through

$$Ju(\phi) = \frac{1}{2}\int Ju \wedge \phi\, dx$$

for ϕ an $(n-2)$-form. This current corresponds to the vorticity lines in 3D. For example if $u = e^{i\phi}$ with singular set γ straight line parallel to the z axis and a degree D around γ, then $Ju = \pi D(dx \wedge dy) \lfloor \gamma$ i.e. given test vector-fields X and Y,

$$Ju(X,Y) = D\int_\gamma \pi\left(\frac{X_1Y_2 - X_2Y_1}{2}\right)$$

The total variation of Ju is then $|Ju| = \pi|D|\mathcal{H}^1(\gamma)$, multiple of the length of the line.

Theorem 7 ([18]) *Let u_ε be a family of $H^1(\Omega,\mathbb{C})$ such that*

$$E_\varepsilon(u_\varepsilon) \le C|\log\,\varepsilon|$$

then up to extraction, $\forall\beta > 0$, $Ju_\varepsilon \rightharpoonup J$ in $(C_C^{0,\beta}(\Omega))^*$, with $\frac{J}{\pi}$ an $(n-2)$-dimensional rectifiable integer-multiplicity current (see [13]) and for every X,Y continuous compactly supported vector fields, we have

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log\,\varepsilon|}\sqrt{\int_\Omega |X\cdot\nabla u_\varepsilon|^2 \int_\Omega |Y\cdot\nabla u_\varepsilon|^2} \ge \left|\int_\Omega J(X,Y)\right|. \tag{24}$$

As a first corollary, in 2D, taking $X = e_1, Y = e_2$ an orthonormal basis, we find

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log\,\varepsilon|}\sqrt{\int_\Omega |\partial_1 u_\varepsilon|^2 \int_\Omega |\partial_2 u_\varepsilon|^2} \ge \pi\sum_i |d_i|$$

This estimate implies the estimate $\varliminf_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} \ge \pi\sum_i |d_i|$ but is sharper. It implies in particular that if $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \le \pi\sum_i |d_i||\log\,\varepsilon|$ and $d_i = \pm 1$, then

$$\forall X, \quad \lim_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|}\int_\Omega |\nabla u_\varepsilon\cdot X|^2 = \pi\sum_i |X(a_i)|^2 \tag{25}$$

i.e. there is isotropy in the repartition of the energy along different directions.

In 3D, taking $X \perp Y$ and maximizing the right-hand side of (24) over $|X|, |Y| \le 1$ we find

$$\varliminf_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} \ge |J|(\Omega),$$

an estimate that was previously proved in [13].

Remark: Our result extends to higher energies $E_\varepsilon(u_\varepsilon) \le N_\varepsilon|\log\varepsilon|$ with N_ε unbounded, in that case we just need to rescale by N_ε and replace J by the limit of $\frac{Ju_\varepsilon}{N_\varepsilon}$.

III.2 Idea of the proof

The method consists in reducing to two dimensions. By using partitions of unity, we can assume that X and Y are locally constant. We may then work in an open set U where X and Y are constant. If they are not parallel, they define a planar direction (if they are then $J(X,Y) = 0$ and there is nothing to prove). We then slice U into planes parallel to that plane. Assume $X = e_1$ and $Y = e_2$ orthonormal vectors. In each plane we have the known 2D lower bounds of the type

$$\frac{1}{2}\int_{\text{plane}\cap U} |\nabla u_\varepsilon|^2 \geq \pi \sum |d_i||\log\, \varepsilon|$$

where d_i is the degree of the boundary of the balls, constructed with the ball-construction method (see [17, 11, 20]). This is possible as long as there is a good bound on the energy on that planar slice, and the number of balls can be unbounded.

The main trick is to observe that this is true for any metric in the plane, and use the metric $\lambda dx + \frac{1}{\lambda} dy$, leading to

$$\frac{1}{2\lambda}\int_{plane\cap U} |\partial_1 u_\varepsilon|^2 + \frac{\lambda}{2}\int_{plane\cap U} |\partial_2 u_\varepsilon|^2 \geq \pi \sum |d_i||\log \varepsilon|$$

Integrating with respect to the slices yields

$$\frac{1}{2\lambda}\int_U |\nabla u_\varepsilon \cdot X|^2 + \frac{\lambda}{2}\int_U |\nabla u_\varepsilon \cdot Y|^2 \geq \left|\int_U J(X,Y)\right| |\log\, \varepsilon|.$$

Optimizing with respect to λ, we conclude that

$$\underline{\lim}_{\varepsilon\to 0} \frac{1}{|\log\, \varepsilon|}\int_U |\nabla u_\varepsilon \cdot X|^2 \int_U |\nabla u_\varepsilon \cdot Y|^2 \geq \left|\int_U J(X,Y)\right|$$

and we may finish by adding these estimates thanks to the partitions of unity.

III.3 Application to the dynamics

In order to deduce a result for the dynamics in 2D, the idea is to use this theorem in dimension $n = 3$ with 2 coordinates corresponding to space coordinates and 1 coordinate corresponding to the time coordinate (this can be done in any dimension, but we restrict to 2D here for the sake of simplicity). The vortex-lines in 3D are then the trajectories in time of the vortex-points in 2D, and clearly the length of these lines is somehow related to the velocity of these points. Splitting the coordinates, we write

$$\begin{aligned} ju \quad &= \langle iu, \partial_t u\rangle\, dt + \langle iu, d_{space}\, u\rangle \\ Ju_\varepsilon \quad &= \underbrace{\sum_{i=1}^{2} V_i\, dt \wedge dx_i}_{V_\varepsilon} + \underbrace{\frac{1}{2} d_{space}\langle iu, d_{space} u\rangle}_{\mu_\varepsilon \to \pi \sum d_i \delta_{a_i(t)}} \end{aligned}$$

Theorem 8 ([18]) *Let $u_\varepsilon(t,x)$ be defined over $[0,T]\times\Omega$ ($\Omega\subset\mathbb{R}^n$, here $n=2$) and such that*

$$\begin{cases} \forall t \quad E_\varepsilon(u_\varepsilon(t)) \le C|\log\,\varepsilon| \\ \displaystyle\int_{[0,T]\times\Omega} |\partial_t u_\varepsilon|^2 \le C|\log\,\varepsilon| \end{cases}$$

then

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } (C_C^{0,\gamma})^*, \qquad \mu \in L^\infty([0,T], C_0^0(\Omega)^*), \quad \mu = \pi\sum_i d_i\,\delta_{a_i(t)}$$

$$V_\varepsilon \rightharpoonup V, \qquad V\in L^\infty([0,T],C_0^0(\Omega)^*)$$

with

$$d_t\mu + \operatorname{div}\ V = 0.$$

Moreover, $\forall X\in C_C^0([0,T]\times\Omega,\mathbb{R}^n)$, and $f\in C_C^0([0,T]\times\Omega)$,

$$\varliminf_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|}\sqrt{\int_{[0,T]\times\Omega}|X\cdot\nabla u_\varepsilon|^2\int_{[0,T]\times\Omega} f^2|\partial_t u_\varepsilon|^2} \ge \left|\int_{[0,T]\times\Omega} V\cdot fX\right|$$

In 2D, the vector $V=(V_1,V_2)$ really is $\pi\sum_i d_i(\partial_t a_i)\delta_{a_i(t)}$, such that

$$\partial_t\left(\pi\sum_i d_i\,\delta_{a_i(t)}\right) + \operatorname{div}\ V = 0$$

Corollary 1 *If in addition $d_i=\pm1$ and $\forall t$, $\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2 \le \pi\left(\sum_i|d_i|\right)|\log\,\varepsilon|(1+o(1))$, then for all intervals $[t_1,t_2)$ on which the a_i's remain distinct, we have*

$$\varliminf_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|}\int_{\Omega\times[t_1,t_2]}|\partial_t u_\varepsilon|^2 \ge \pi\sum_i\int_{t_1}^{t_2}|\partial_t a_i|^2\,dt$$

This is the desired estimate 1) in Theorem 5. To prove this corollary, recall that from (25), if $E_\varepsilon(u_\varepsilon)\simeq \pi n|\log\,\varepsilon|$ then $\frac{1}{|\log\,\varepsilon|}\int_\Omega|X\cdot\nabla u_\varepsilon|^2 \simeq \pi\sum_i|X(a_i)|^2$ and optimizing over X and f gives the L^2 bound on V.

III.4 Proof of the construction 2′)

We wish to prove that 2') holds for Ginzburg-Landau so that we deduce 2) i.e. if $u_\varepsilon \overset{S}{\rightharpoonup} u$ then $\varliminf_{\varepsilon\to0}\|\nabla_{X_\varepsilon}E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \ge \|\nabla W(u)\|_Y$.

Observe that this is a static result. We thus assume that curl $\langle iu_\varepsilon,\nabla u_\varepsilon\rangle \rightharpoonup 2\pi\sum_i d_i\delta_{a_i}$, where $u=((a_1,d_1),\cdots,(a_n,d_n))$ and may consider disjoint balls $B(a_i,\rho)$ of fixed radius ρ. If $\|\nabla_{X_\varepsilon}E(u_\varepsilon)\|_{X_\varepsilon}\to+\infty$ there is nothing to prove. If $\|\nabla_{X_\varepsilon}E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}=O(1)$ then we can prove that $D_\varepsilon=o(1)$ where $D_\varepsilon=E_\varepsilon(u_\varepsilon)-\pi n|\log\,\varepsilon|-W(u)$ is the "energy-excess". The proof of this result (see [22, 21]) relies on the fact that $\|\nabla_{X_\varepsilon}E_\varepsilon(u)\|_{X_\varepsilon}\le C$ means $\int_\Omega\left|\Delta u+\frac{u}{\varepsilon^2}\left(1-|u|^2\right)\right|^2 \le$

$\frac{C}{|\log\,\varepsilon|} = o(1)$ and one can take advantage of the fact that u_ε is thus an "almost-solution".

Once this is proved, we may deduce

$$\frac{1}{2}\int_{B(a_i,\rho)} |\nabla u_\varepsilon|^2 = \pi|\log\,\varepsilon| + O(1)$$

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla|u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le D_\varepsilon = o(1) \tag{26}$$

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} \left|\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\Phi_0\right|^2 \le D_\varepsilon = o(1) \tag{27}$$

where

$$\Delta\Phi_0 = 2\pi\sum_i d_i\delta_{a_i} \quad \text{in } \Omega$$

with the appropriate boundary conditions. The rough idea is that $\nabla\varphi_\varepsilon \simeq \nabla^\perp\Phi_0$ outside of the vortex balls. Through these relations, everything is well-controlled outside the balls and inside the balls we shall only perform a pure translation.

Given $V = (V_1,\dots,V_n)$, we want to push each a_i in the direction V_i. For that purpose, define $\chi_t(x) = x + tV_i$ in each B_i, and extend it in a smooth way outside of the B_i's into a family of smooth diffeomorphisms that keep $\partial\Omega$ fixed and are *independent of* ε. Choosing the deformation

$$v_\varepsilon(x,t) = u_\varepsilon(\chi_t^{-1}(x))$$

does the job of pushing the vortices a_i along the direction V_i. However it is not enough, and we need to add a phase correction ψ_t:

$$v_\varepsilon\left(\chi_t(u),t\right) = u_\varepsilon(x)\,e^{i\psi_t(x)} \tag{28}$$

so that for every t, the phase of v_ε is approximately the optimal one, that is the harmonic conjugate of

$$\Delta\Phi_t = 2\pi\sum_i d_i\,\delta_{a_i(t)} \quad a_i(t) = a_i + tV_i.$$

It is possible to construct ψ_t single-valued, independent of ε, so that

$$\nabla^\perp\Phi_0 + \nabla\psi_t \simeq \nabla^\perp\left(\Phi_t\circ\chi_t\right).$$

We will now check that the v_ε constructed this way works. First,

$$\begin{aligned}\frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) &\simeq \frac{1}{|\log\,\varepsilon|}\sum_i\int_{B_i}|V_i\cdot\nabla u_\varepsilon|^2 + o(1)\\ &\simeq \pi\sum_i|V_i|^2 + o(1)\end{aligned}$$

because χ_t achieves a translation of vector V_i in the B_i's while the contribution outside of the B_i's is negligible; and from the relation (25). The first requirement for 2') is thus fulfilled. Let us check the second requirement, i.e. the energy-decrease rate, by evaluating $\frac{d}{dt}_{|t=0} E_\varepsilon\left(v_\varepsilon(t)\right)$. With a change of variables,

$$\begin{aligned} E_\varepsilon\left(v_\varepsilon(t)\right) &= \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(1-|v_\varepsilon|^2\right)^2 \\ &= \frac{1}{2}\int_\Omega \left(\left|D\chi_t^{-1}\,\nabla(v_\varepsilon\circ\chi_t)\right|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2\right)|Jac\ \chi_t|. \end{aligned}$$

Now, recall that χ_t is a translation in $\cup_i B_i$ hence $|Jac\ \chi_t| = cst$ there, while outside of $\cup_i B_i$ there is almost no energy, hence

$$\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0}\int_\Omega \left|D\,\chi_t^{-1}\,\nabla\left(u_\varepsilon\, e^{i\psi_t}\right)\right|^2\ |Jac\ \chi_t| + o(1).$$

Next, we expand $\nabla\left(u_\varepsilon\, e^{i\psi_t}\right)$ as $\nabla u_\varepsilon e^{i\psi_t} + i u_\varepsilon \nabla\psi_t$, expand the squares, and apply $\frac{d}{dt}_{|t=0}$. The crucial fact is that the terms which get differentiated do not depend on ε. For the other terms, we use (27), so that there remains

$$\begin{aligned} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) &= \int_{\Omega\setminus\cup_i B_i}\left(\frac{d}{dt}_{|t=0} D\chi_t^{-1}\right)\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0 \\ &\quad + \int_{\Omega\setminus\cup_i B_i}\frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla^\perp\Phi_0 + \frac{1}{2}|\nabla\Phi_0|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t| + o(1) \\ &= \frac{d}{dt}_{|t=0}\frac{1}{2}\int_{\Omega\setminus\cup_i B_i}|D\chi_t^{-1}(\nabla^\perp\Phi_0 + \nabla\psi_t)|^2|Jac\ \chi_t| + o(1) \end{aligned}$$

But observing that ψ_t was constructed in such a way that $\nabla^\perp\Phi_0 + \nabla\psi_t = \nabla^\perp(\Phi_t\circ\chi_t)$, and doing a change of variables again, we find

$$\begin{aligned} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) &= \frac{d}{dt}_{|t=0}\frac{1}{2}\int_{\Omega\setminus\cup_i B_i}|\nabla\Phi_t|^2 + o(1) \\ &= \frac{d}{dt}_{|t=0} W_{\mathbf{d}}(a_1(t),\cdots,a_n(t)) + o(1) \end{aligned}$$

i.e. the desired result.

IV Extensions of the method

IV.1 Collisions

When there are some positive as well as some negative vortices, the limiting dynamics (22) induces collisions between vortices of opposite signs which attract each other. One expects that those vortices annihilate and the dynamics continues with the remaining ones. This has been treated recently in the papers [4, 5, 21].

One of the things done in [21] was to extend the method presented here to treat the case of collisions. If two vortices of opposite degrees get at a distance $l_\varepsilon = o(1)$ of each other, then it is possible to rescale space in order to have two vortices at distance 1. As long as $|\log l_\varepsilon| \ll |\log \varepsilon|$ the same method we presented above carries through, i.e. we can prove the analogues of conditions 1) and 2), and yields that the dynamics continues with the same type of law (22) but in space-time rescaled coordinates.

When vortices become too close to apply this, we focused on evaluating energy dissipation rates, through the study of the perturbed Ginzburg-Landau equation

$$\Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2) = f_\varepsilon \quad \text{in } \Omega, \tag{29}$$

with Dirichlet or Neumann boundary data, where f_ε is given in $L^2(\Omega)$ (recall the instantaneous energy-dissipation rate in the dynamics is exactly $|\log \varepsilon| \|f_\varepsilon\|^2_{L^2(\Omega)}$). We prove that the energy-excess (still meaning the difference between $E_\varepsilon - \pi n |\log \varepsilon|$ and the renormalized energy W of the underlying vortices) is essentially controlled by $C\|f_\varepsilon\|^2_{L^2}$. We then show that when u solves (29) and has vortices which become very close, forming what we call an "unbalanced cluster" in the sense that $\sum_i d_i^2 \neq (\sum_i d_i)^2$ in the cluster (see [21] for a precise definition), then the lower bound

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \geq \min\left(\frac{C}{l^2|\log \varepsilon|}, \frac{C}{l^2 \log^2 l}\right) \tag{30}$$

holds. In particular, when vortices get close to each other, say two vortices of opposite degrees for example, then they form an unbalanced cluster of vortices at scale $l =$ their distance, and the relation (30) gives a large energy-dissipation rate (scaling like $1/l^2$). This serves to show that such a situation cannot persist for a long time and we are able to prove that the vortices collide and disappear in time $Cl^2 + o(1)$, with all energy-excess dissipating in that time. Thus after this time $o(1)$, the configuration is again "well-prepared" and Theorem 6 can be applied again, yielding the dynamical law with the remaining vortices, until the next collision, etc...

IV.2 Second order questions - stability issues

We extended the "Γ-convergence" method to second order in order to treat stability questions for this 2D Ginzburg-Landau equation. Here is the abstract result, pushing the method of condition 2') to second order. The setting is as in Section II.1. By stable critical point, we mean nonnegative Hessian.

Theorem 9 ([22]) *Let u_ε be a family of critical points of E_ε with $u_\varepsilon \rightharpoonup^S u \in \mathcal{N}$, such that the following holds: for any $V \in \mathcal{B}'$, we can find $v_\varepsilon(t) \in \mathcal{M}$ defined in a neighborhood of $t = 0$, such that $\partial_t v_\varepsilon(0)$ depends on V in a linear* *and one-to-one manner, and*

$$v_\varepsilon(0) = u_\varepsilon(0) \tag{31}$$

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0} F(u+tV) = dF(u).V \tag{32}$$

$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d^2}{dt^2}_{|t=0} F(u+tV) = Q(u)(V). \tag{33}$$

Then
- if (31)-(32) are satisfied, then u is a critical point of F
- if (31)-(32)-(33) are satisfied, then if u_ε are stable critical points of E_ε, u is a stable critical point of F. More generally, denoting by n_ε^+ the dimension (possibly infinite) of the space spanned by eigenvectors of $D^2E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues, and n^+ the dimension of the space spanned by eigenvectors of $D^2F(u)$ associated to positive eigenvalues (resp. n_ε^- and n^- for negative eigenvalues); for ε small enough we have

$$n_\varepsilon^+ \geq n^+ \qquad n_\varepsilon^- \geq n^-.$$

Thus, we reobtain that critical points converge to critical points of the limiting energy F (proved in [2] for Ginzburg-Landau), but in addition we obtain that under certain conditions, stability/instability of the critical point also passes to the limit. The previous result (Theorem 1), was an analysis of the C^1 structure of the energy landscape, thus suited to give convergence of gradient-flow and critical points; while this is the C^2 analysis of the energy landscape around a critical point.

For the Ginzburg-Landau energy, the construction done in Section III.4 can be pushed to second order, yielding condition (33). Thus we deduce the corresponding theorem for solutions of (3), (see [22]). An interesting application is for Neumann boundary condition, for which it is known that the corresponding renormalized energy W has *no stable critical point.* Hence from Theorem 9 there can be no stable critical points of E_ε with vortices (in contrast with the case of G_ε with nonzero applied magnetic field).

Theorem 10 ([22]) *Let u_ε be a family of nonconstant solutions of*

$$\begin{cases} -\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

(on $\Omega \subset \mathbb{R}^2$ simply connected) such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$; then, for ε small enough, u_ε is unstable.

This is an extension of a result of Jimbo and Sternberg [14] for convex domains.

References

[1] Ambrosio, L.; Gigli, N.; Savarè, G. *Gradient flows in metric spaces and in the Wasserstein spaces of probability measures*, Birkhaüser, 2005.

[2] Bethuel, F.; Brezis, H.; Hélein, F. *Ginzburg-Landau vortices.* Progress in Nonlinear Partial Differential Equations and Their Applications, 13. Birkhaüser Boston, Boston, 1994.

[3] Bethuel, F.; Brezis, H.; Orlandi, G. Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions. *J. Funct. Anal.* **186** (2001), no. 2, 432–520.

[4] Bethuel, F.; Orlandi, G.; Smets, D. Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics. *Duke Math. J.*, to appear.

[5] Bethuel, F.; Orlandi, G.; Smets, D. Quantization and motion laws for Ginzburg-Landau vortices. Preprint, 2005.

[6] Chen, X. Generation and propagation of interfaces for reaction-diffusion equations, *J. Diff. Eq.* **96** (1992), 116-141.

[7] DeGennes, P. G. *Superconductivity of metal and alloys.* Benjamin, New York and Amsterdam, 1966.

[8] De Mottoni, P; Schatzman, M. Development of interfaces in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh, Sec A* **116** (1990), 207-220.

[9] Evans, L.C; Soner, H.M; Souganidis, P.E. Phase transitions and generalized motion by mean curvature, *Comm. Pure Appl. Math.* **45**, No 9, (1992), 1097–1123.

[10] Ilmanen, T. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, *J. Diff. Geom.* **38** (1993), 417-461.

[11] Jerrard, R. L. Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* **30** (1999), no. 4, 721-746.

[12] Jerrard, R. L.; Soner, H. M. Dynamics of Ginzburg-Landau vortices. *Arch. Ration. Mech. Anal.* **142** (1998), no. 2, 99–125.

[13] Jerrard, R. L.; Soner, H. M. The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations* **14** (2002), no. 2, 151–191.

[14] Jimbo, S.; Sternberg, P. Nonexistence of permanent currents in convex planar samples. *SIAM J. Math. Anal.* **33** (2002), no. 6, 1379–1392.

[15] Lin, F.-H. Some dynamic properties of Ginzburg-Landau vortices. *Comm. Pure Appl. Math* **49** (1996), 323–359.

[16] Lin, F.-H.; Rivière, T. A quantization property for static Ginzburg-Landau vortices. *Comm. Pure Appl. Math.* **54** (2001), no. 2, 206–228.

[17] Sandier, E. Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998), no. 2, 379–403; Erratum, Ibid, **171** (2000), no. 1, 233.

[18] Sandier, E.; Serfaty, S. A product-estimate for Ginzburg-Landau and corollaries. *J. Funct. Anal.* **211** (2004), no. 1, 219–244.

[19] Sandier, E.; Serfaty, S. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.* **57** (2004), no. 12, 1627–1672.

[20] Sandier, E.; Serfaty, S. *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhaüser, to appear.

[21] Serfaty, S. Vortex-collision and energy dissipation rates in the Ginzburg-Landau heat flow, part I: Study of the perturbed Ginzburg-Landau equation; part II: The dynamics. Preprint, 2005.

[22] Serfaty, S. Stability in 2D Ginzburg-Landau passes to the limit. *Indiana Univ. Math. J* **54** (2005), no. 1, 199–222.

[23] Spirn, D. Vortex motion law for the Schrdinger-Ginzburg-Landau equations. *SIAM J. Math. Anal.* **34** (2003), no. 6, 1435–1476.

[24] Tinkham, M. *Introduction to superconductivity.* Second edition. McGraw-Hill, New York, 1996.

The author wishes to thank warmly the organizers of the mini-course, and in particular Andrea Braides, for their invitation and hospitality.