Divergence-conforming multigrid methods for incompressible flow problems

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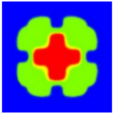

Prague-Heidelberg-Workshop
April 28th, 2015

Stokes and Navier-Stokes flow

- Note: all equations in weak form
- Stationary/transient Navier - Stokes equations

$$(\partial_t u, v) + 2\mu(Du, Dv) + (u \cdot \nabla u) + (\nabla \cdot v, p) + (\nabla \cdot u, q) = (f, v)$$

  - $2Du = \nabla u + \nabla u^T$ is the symmetric gradient

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- Spaces:
  - $H^1(\Omega; \mathbb{R}^d)$ for velocities u, v
  - $L^2(\Omega)$ for pressures p, q
  - Restriction by suitable boundary conditions

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- Spaces:
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- Simplified stationary Navier - Stokes equations

$$\nu(\nabla u, \nabla v) + (u \cdot \nabla u) + (\nabla \cdot v, p) + (\nabla \cdot u, q) = (f, v)$$

- Other forms of Navier term possible (“symmetric”, conservative)

Darcy flow

- Darcy equations

$$(K^{-1}u, v) + +(\nabla \cdot v, p) + (\nabla \cdot u, q) = (f, v)$$

- Permeability or permeability tensor K.

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Abstract incompressible flow

- Common form of examples:

$$a(u,v) + (\nabla\cdot v, p) + (\nabla\cdot u, q) = (f, v)$$

- Suitable subspaces $V \subset H^{\text{div}}$ for velocities u, v and $Q \subset L^2$ for pressures p, q.
- V, Q determined by boundary conditions
- Bilinearform $a(.,.)$ stable and bounded on divergence free subspace of V

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Abstract incompressible flow

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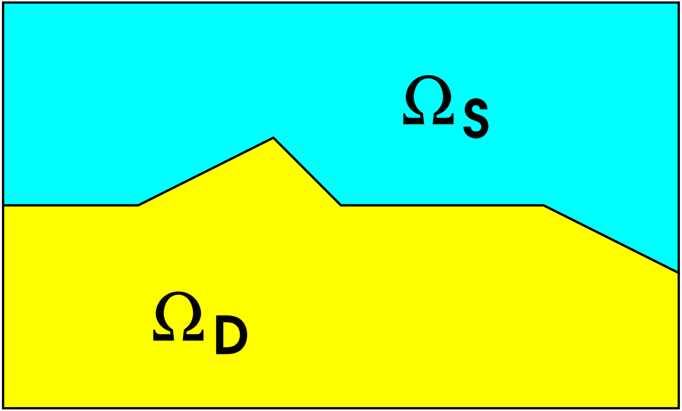
- V, Q determined by boundary conditions
- Bilinearform $a(.,.)$ stable and bounded on divergence free subspace of V
- These conditions imply solvability in some sense
  - Unique, stable solutions for linear problems
  - But, beware of Navier-Stokes

Darcy-Stokes coupling

- Stokes in subdomain Ω_S, Darcy in Ω_D.



Darcy-Stokes coupling

- Stokes in subdomain Ω_S, Darcy in Ω_D.
- Interface conditions (Beavers-Joseph-Saffman)
  - Mass conservation

$$u_n^S = u_n^D$$

  - Balance of forces

$$p^S - \nu \partial_n u_n^S = p^D$$

  - Friction condition

$$\nu \partial_n u_\tau^S - \gamma K^{-1/2} u_\tau^S = 0$$

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Weak formulation of Darcy-Stokes coupling

- Mixed bilinear form

$$\left(K^{-1}u, v\right)_{\Omega_D} + 2\mu\left(Du, Dv\right)_{\Omega_S} + \left(\gamma K^{-1/2}u_\tau^S, v_\tau^S\right)_\Gamma \\ + \left(\nabla\cdot v, p\right) + \left(\nabla\cdot u, q\right) = (f, v)$$

- Velocity space

$$V = \left\{v \in H^{\text{div}}(\Omega) \middle| v|_{\Omega_S} \in H^1(\Omega_S;\mathbb{R}^d) \text{ and b. c.}\right\}$$

- Conservative at the interface

The Hilbert cochain complex

- Exact sequence in 3D

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\nabla} & H^{\text{curl}}(\Omega) & \xrightarrow{\nabla\times} & H^{\text{div}}(\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) & \longrightarrow & \mathbb{R} \\
 & & \text{b.c.}\downarrow & & \text{b.c.}\downarrow & & \text{b.c.}\downarrow & & \downarrow & & \\
 & & X & \xrightarrow{\nabla} & \Psi & \xrightarrow{\nabla\times} & V & \xrightarrow{\nabla\cdot} & Q & &
\end{array}
$$

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- Exact sequence in 2D

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\nabla\times} & H^{\mathrm{div}}(\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) & \longrightarrow & \mathbb{R} \\
 & & \text{b.c.}\downarrow & & \text{b.c.}\downarrow & & \downarrow & & \\
 & & \Psi & \xrightarrow{\nabla\times} & V & \xrightarrow{\nabla\cdot} & Q & &
\end{array}$$

Use of the cochain complex

- Hodge decomposition

$$V^0 := \ker(\nabla\cdot) = \mathrm{im}(\nabla\times) \oplus \mathcal{H}$$

- All subspaces are closed (continuous projectors)
- Harmonic forms $\mathcal{H}$ are finite dimensional
- Can we mimick this with finite elements?

Finite element cochain complexes

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\nabla} & H^{\mathrm{curl}}(\Omega) & \xrightarrow{\nabla\times} & H^{\mathrm{div}}(\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) & \longrightarrow & \mathbb{R} \\
 & & \Pi_h \downarrow & & \Pi_h \downarrow & & \Pi_h \downarrow & & \Pi_h \downarrow & & \\
 & & Q_{k+1}^{\mathrm{conf}} & \xrightarrow{\nabla} & N_k & \xrightarrow{\nabla\times} & RT_k & \xrightarrow{\nabla\cdot} & Q_k^{\mathrm{dg}} & &
\end{array}$$

$Q_{k+1}^{\mathbf{conf}}$**:** H^1-conforming tensor product polynomials

N_k**:** Nedelec elements

RT_k**:** Raviart-Thomas elements

$Q_k^{\mathbf{dg}}$**:** discontinuous tensor product polynomials

Other options based on BDM or on simplices

Finite element cochain complex in 2D

$$
\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & \Psi & \xrightarrow{\nabla\times} & V & \xrightarrow{\nabla\cdot} & Q & \longrightarrow & \mathbb{R} \\
 & & \Pi_h \big\downarrow & & \Pi_h \big\downarrow & & \Pi_h \big\downarrow & & \\
 & & \Psi_h & \xrightarrow{\nabla\times} & V_h & \xrightarrow{\nabla\cdot} & Q_h & & \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
 & & Q_{k+1}^{\text{conf}} & & RT_k & & Q_k^{\text{dg}} & &
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Raviart-Thomas elements for flow

- Subspace of H^{div}, therefore conforming for Darcy
- Not a subspace of H^1, therefore inconsistent for Stokes
- Possible solution: higher continuity
  - Conforming subspace
  - Difficult to achieve, but exists
  - Restriction on meshes

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  - Conforming subspace
  - Difficult to achieve, but exists
  - Restriction on meshes
- Possible solution: consistency through discontinuous Galerkin
  - A DG formulation for the Laplacian is needed

DG-Stokes cochain complex: **K.**/Sharma

DG step 1: Nitsche boundary conditions

- Goal: solution of the Dirichlet problem for the Laplacian
- Finite element space does not obey boundary conditions
    - Inconsistency (weak form solves Neumann problem)

$$(\nabla u, \nabla v)_\Omega$$

$$= (f, v)_\Omega$$

DG step 1: Nitsche boundary conditions

- Goal: solution of the Dirichlet problem for the Laplacian
- Finite element space does not obey boundary conditions
  - Inconsistency (weak form solves Neumann problem)
- Remove natural boundary condition

$$
\begin{aligned}
(\nabla u, \nabla v)_\Omega &- (\partial_n u, v)_{\partial\Omega} \\
&= (f, v)_\Omega
\end{aligned}
$$

DG step 1: Nitsche boundary conditions

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- Symmetrize the bilinear form
- Additional terms are indefinite

$$
\begin{aligned}
(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_{\partial\Omega} - (u, \partial_n v)_{\partial\Omega} \\
= (f, v)_\Omega - (u^D, \partial_n v)_{\partial\Omega}
\end{aligned}
$$

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- Finite element space does not obey boundary conditions
  - Inconsistency (weak form solves Neumann problem)
- Remove natural boundary condition
- Symmetrize the bilinear form
- Additional terms are indefinite
- Stabilize

$$
\begin{aligned}
(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_{\partial\Omega} - (u, \partial_n v)_{\partial\Omega} + \tfrac{\kappa}{h}(u, v)_{\partial\Omega} \\
= (f, v)_\Omega - (u^D, \partial_n v)_{\partial\Omega} + \tfrac{\kappa}{h}(u^D, v)_{\partial\Omega}
\end{aligned}
$$

DG step 2: interior penalty

1. Apply Nitsche's method on each cell of the mesh $\mathbb{T}_h$, functions not continuous at element faces
2. Do some reshuffling on interior faces
   - Replace $\partial_n u$ by averages $\{\!\{\nabla u\}\!\}$ from left and right
   - Now every $\{\!\{\nabla u\}\!\}$ appears twice with different test functions:

$$\int_F 2\{\!\{\nabla u\}\!\}\{\!\{v\mathbf{n}\}\!\}\,ds$$

   - $\{\!\{v\mathbf{n}\}\!\}$ is a jump
3. Add consistent stabilization

$$\int_F \frac{\kappa}{h}\{\!\{u\mathbf{n}\}\!\}\{\!\{v\mathbf{n}\}\!\}\,ds$$

$\Rightarrow$ Interior penalty method (Arnold, Wheeler, ...)

DG step 3: other schemes (Laplacian)

- Babuška/Zlámál, Baker, Interior Penalty, Bassi/Rebai

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1. Consistent

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...

- Reduction to three assumptions

1. Consistent
2. Adjoint consistent

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...

- Reduction to three assumptions

1. Consistent
2. Adjoint consistent
3. Stable and bounded in the broken H^1-norm

$$\|v\|_{1,h}^2 = \sum_{T\in\mathbb{T}_h} |v|_{1;T}^2 + \sum_{F\in\mathbb{F}_h} \frac{1}{h_F} \|\{\!\{v\mathbf{n}\}\!\}\|_F^2$$

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Personal opinion: Interior penalty is simple and has all good properties

Summary discretization

- Darcy/Stokes coupling
- Beavers/Joseph/Saffman condition
- Velocities in H^{div} space based on RT_k
- Interior penalty for consistency with H^1

Convergence theorems

1. Pointwise divergence free velocities or (Cockburn/**K.**/Schötzau, **K.**/Rivière)

$$\nabla\cdot u_h = \Pi_{L^2\to Q_h}\nabla\cdot u \qquad \|\nabla\cdot u - \nabla\cdot u_h\| = \mathcal{O}(h^{k+1})$$

2. Balanced L^2 approximation (Girault/**K.**/Rivière)

$$\|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{k+1})$$

Benefits from cochain property

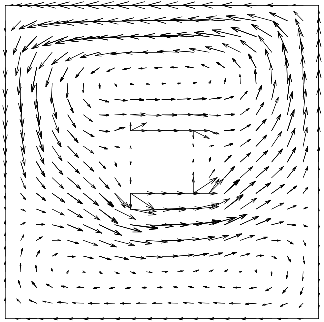
- Pressure free a posteriori estimates derived from cochain complex

$$\|u - u_h\|_A \leq \eta(u_h, f^0)$$

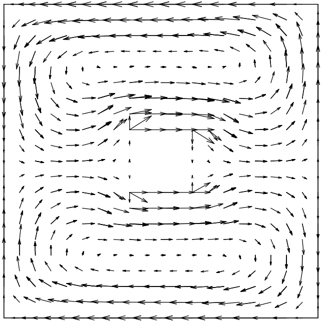
- Examples:
  - boundary driven flows
  - gravitation forces
- Adaptive iteration on velocity space only
- Relation to C^0-IP method
- Multigrid

Sharma/**K.**

Why not stream function?



u_h



$\nabla \times \psi_h$

- Global Hodge decomposition fails on not simply connected domains
- Analysis and multigrid only use local decompositions

Solution of the discrete systems

Saddle point system for u and p

1. Schur complement iterations
   - Generally slow
   - No closed form of pressure Schur complement
2. Alternating solvers on subdomains
   - Relies on simple domain structure
   - Particular solution, not generic
3. Multigrid for the whole system
   - Generic solution for all cochain problems
   - Implicitly solves in divergence free subspace

Hierarchies of level spaces

- Sequence of meshes through refinement

$$\mathbb{T}_0 \sqsubset \mathbb{T}_1 \sqsubset \cdots \sqsubset \mathbb{T}_l \sqsubset \cdots \sqsubset \mathbb{T}_L$$

- Sequences of nested spaces

$$V_0 \subset V_1 \subset \cdots \subset V_l \subset \cdots \subset V_L$$
$$Q_0 \subset Q_1 \subset \cdots \subset Q_l \subset \cdots \subset Q_L$$

- Subspace property defines canonical embedding operators

$$I_{V,\ell} : V_{\ell-1} \to V_\ell \qquad I_{Q,\ell} : Q_{\ell-1} \to Q_\ell$$

- The index ℓ in the hierarchy is called “level”

The multigrid V-cycle

The multigrid V-cycle: given a current guess, do iteratively:

1. Perform one or more pre-smoothing steps on the current level
2. Coarse grid correction:
   1. Project the residual to the next coarser level
   2. Perform one step of this algorithm there (or solve on the coarsest level)
   3. Add the result to the pre-smoothened solution
3. Perform one or more post-smoothing steps on the current level

If on the coarsest level, solve exactly.

Construction of divergence free multigrid

Simple guiding principles:

- The smoother may never produce a solution which is not divergence free
- The result of the coarse grid correction must be divergence free

Tools:

- Subspace correction methods (Schwarz)
- Cochain complexes

Hierarchy of nested complexes

$$
\begin{array}{ccccc}
I_{\Psi,\ell+1}\Big\uparrow & & I_{V,\ell+1}\Big\uparrow & & I_{Q,\ell+1}\Big\uparrow \\
\Psi_\ell & \xrightarrow{\nabla\times} & V_\ell & \xrightarrow{\nabla\cdot} & Q_\ell \\
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\end{array}
$$

Consequence: divergence free subspaces are nested as well!

$$I_{V,\ell} : V^0_{\ell-1} \to V^0_\ell$$

Coarse grid corrections are divergence free, if smoother results are

Smoothers

Original idea: Arnold/Falk/Winther 1997

- On level ℓ, find many small subspaces $V_{\ell,i}$
  - Have a commuting diagram for embedding operators
  - Sum of divergence free subspaces is big enough

$$\sum V^0_{\ell,i} = V^0_\ell$$

  - Typically: patches of cells around vertices
  - Solve Stokes problems in $V_{\ell,i}$ and add or compose
  - Result will be divergence free

Multigrid summary

Contraction theorems

The described multigrid method is a uniform contraction independent of the level L.

- Darcy with smooth permeability: Arnold/Falk/Winther, Powell, Schöberl
- Stokes: **K.**/Mao

Missing analysis:

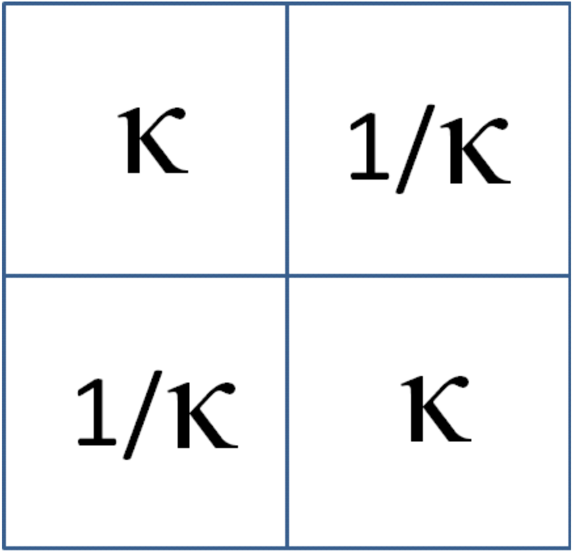
- Darcy with high contrast
- Darcy/Stokes coupling

Some results for Stokes

level	RT_1	RT_2	RT_3
2	1	1	1
3	3	3	3
4	5	5	5
5	6	6	6
6	6	6	6
7	6	6	6
8	7	6	6

Number of iterations n_{10} to reduce the residual by 10^{-10},

Darcy with high contrast



Some results for Darcy with high contrast

Problem with coefficients K and $1/K$, jumps respected by the coarse mesh.

level	10^2	10^3	10^4
2	1	1	1
3	5	3	3
4	5	5	5
5	6	5	5
6	6	6	6
7	6	7	6
8	6	7	7

Preconditioned GMRES steps to gain 10^{10}, symmetric multiplicative method

Some results for Darcy-Stokes

level	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
2	16	16	15	16	18
3	19	18	17	20	23
4	21	21	19	22	27
5	23	23	21	22	29
6	23	23	22	21	28
7	23	23	23	21	26
8	24	23	23	22	24

Conclusions

- Cochain complexes provide a framework for the discretization of various incompressible flow models
- Finite element cochains yield strongly divergence free approximations
- They provide a structure that allows highly efficient multigrid methods

Work in progress:

- Convergent adaptive iterations (w. N. Sharma)
- Navier-Stokes (w. N. Shakir)
- Poroelasticity (w. B. Rivière)
- Fluid structure interaction (L. Heltai)
- Radiation transport (w. J. Ragusa, J. P. Lucero Lorca)

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- Maintained with T. Heister (Clemson) and W. Bangerth (Texas A&M)
- Important additions by contributors
- It's free and LGPL

www.dealii.org