Minimization problems on orientation-preserving bi-Lipschitz maps

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Hyperelasticity

Assumption: 1st Piola-Kirchhoff stress tensor T has a potential:

$$T_{ij} := \frac{\partial W(\nabla y)}{\partial F_{ij}}$$

$W : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{+\infty\}$ stored energy density

$$J(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x \; .$$

Minimizers of J (formally) satisfy equilibrium equations of elasticity.

Properties of W

(i) $W : \mathbb{R}_+^{3\times3} \to \mathbb{R}$ is continuous

(ii) $W(F) = W(RF)$ for all $R \in \mathrm{SO}(3)$ and all $F \in \mathbb{R}^{3\times3}$

(iii) $W(F) \to +\infty$ if $\det F \to 0_+$

(iv) $W(F) = +\infty$ if $\det F \leq 0$

Polyconvexity

J.M. Ball's notion of **polyconvexity** (1977)

$$W(F) = h(F, \operatorname{cof} F, \det F) \text{ if } \det F > 0$$

$$\operatorname{cof} F := (\det F)F^{-\top}$$

$$h : \mathbb{R}^{19} \to \mathbb{R} \text{ is } \textbf{convex}$$

Existence of (injective) solutions (Ball 1977, Ciarlet & Nečas 1987,...)

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How about if W is not polyconvex?

If

$$c(-1+|F|^p) \leq W(F) \leq C(1+|F|^p)$$

and

$$W(F)|\Omega| \leq \int_\Omega W(\nabla\varphi(x))\,dx$$

for all $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^3)$, $\varphi(x) = Fx$ on $\partial\Omega$ then J is wlsc on $W^{1,p}$, $(p>1)$

But the upper bound is not suitable for elasticity !!

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Reconsidering elasticity

- Stable states in elasticity are found through

$$\left.\begin{array}{ll}\text{Minimize} & \int_\Omega W(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in \mathcal{A} = \text{set of deformations.}\end{array}\right\} \quad (1)$$

The energy density W and the set of deformations $\mathcal{A}$ form together the model of the elastic behavior.

What we want to do is the following:

- Characterize precisely the set of energies for which stable states exist
- This is motivated by providing a "safe set" in constitutive modeling

The set of deformations

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A deformation is a mapping $\bar{\Omega} \mapsto \mathbb{R}^3$ that is smooth enough, injective except perhaps on the boundary, and orientation preserving.

[Ciarlet; 1988]

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- this corresponds to non-interpenetration of matter

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[Ciarlet; 1988]

- to exclude "going through itself/reflection"

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A deformation is a mapping that is smooth enough, injective except perhaps on the boundary and orientation preserving.

[Ciarlet; 1988]

$\rightsquigarrow$ these conditions are sometimes written as

$$\text{“}\det \nabla y > 0\text{”}$$

Open problems in elasticity

Providing (at least) that the deformations satisfy $\det\nabla y > 0$ we ask

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- *this is actually a long-standing problem in mathematical elasticity and has been formulated in a related way by J.M. Ball*

Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying

$$W(A) \to +\infty \qquad \text{whenever } \det A \to 0_+$$

[Ball; 2002]

Open problems in elasticity

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Why is this a hard problem?

- the injectivity condition on the deformation is strongly non-linear and non-convex (even in the weakened "determinant condition")
- standard methods relying on convex averaging in Sobolev spaces (such as smoothing by mollifier kernels) do not work

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Elastic deformations

Elasticity means that the specimen returns to its original state when releasing all loads. No energy loss!

No defects in the specimen

- How smooth is smooth enough?
  1. Since the *deformation gradient* is the crucial quantity it is natural to work with Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^n)$
  2. y is continuous
  3. In our modelling, we shall require that *the inverse of the deformation y^{-1} is in the same class as the deformation itself*

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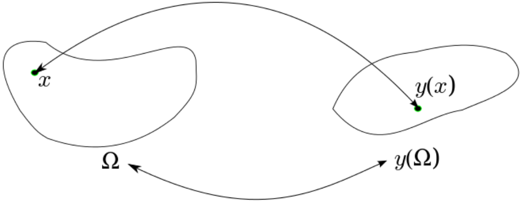
Motivation: The body is deformed from Ω to $y(\Omega)$. Then we change the reference configuration to $y(\Omega)$ and want that the mapping that moves each material point to its original position is an admissible deformation, too.



$\rightsquigarrow$ In a way, we may relate it to *reversibility* of elastic processes (by the same path).

[Giaquinta, Modica, Souček; 1998], [Fonseca, Gangbo; 1995], [Ball; 1981], [Ciarlet, Nečas; 1985], [Šverák; 1988], [Ball, 2002], [Iwaniec, Kovalev, Onninen; 2011]

Deformations in elasticity-Summary

Taking also the preserving of the orientation into account, we take the set of the deformations as the bi-Sobolev maps

$$W^{1,p,-p}(\Omega;\mathbb{R}^3) = \{y; y \text{ homeomorphism, } y \in W^{1,p}(\Omega;\mathbb{R}^3),\ y^{-1} \in W^{1,p}(y(\Omega);\Omega) \text{ and } \det \nabla y > 0 \text{ a.e. on } \Omega\ \}$$

- Let us note that the constraint $\det \nabla y \geq 0$ is "included" when demanding a deformation to be a Sobolev homeomorphism since such cannot change the sign of the determinant in dimension 2,3

[Hencl, Malý; 2010]

Group structure of the set of deformations

The set of the deformations may have a group structure. This models that

- A composition of two deformations is again a deformation.
- Take two deformations $y : \Omega \mapsto \mathbb{R}^3$ and $z : y(\Omega) \mapsto \mathbb{R}^3$. Then the composition of these two deformations is

$$z(y(x)), \qquad x \in \Omega$$

- But the relevant variable is actually the *deformation gradient*

$$\nabla z(y(x)) \nabla y(x)$$

$\rightsquigarrow$ "multiplication should be allowed"

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$\rightsquigarrow$ "multiplication should be allowed"

$\rightsquigarrow$ this is possible only on particular bi-Sobolev classes as e.g. *bi-Lipschitz maps*

The stored energy

On the stored energy we have only two key requirements:

1. W is continuous on its effective domain
2. W has growth that prevents shrinking of volume of positive measure to zero

$$W(A) \to +\infty \qquad \text{whenever } \det A \to 0_+ \tag{2}$$

$\rightsquigarrow$ in some situations, we may prescribe some growth etc.

Posing the problem

$$\left.\begin{array}{ll} \text{Minimize} & \int_\Omega W(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in \mathcal{A}. \end{array}\right\}$$

with

$$\mathcal{A} = W^{1,p,-p}(\Omega; \mathbb{R}^3)$$

an W satisfying

$$W(A) \to +\infty \qquad \text{whenever } \det A \to 0_+$$

Under which (minimal) additional conditions on the stored energy W there is a solution?

Posing the problem

We shall concentrate only on the case when $p = \infty$, i.e.

$$\mathcal{A} = W^{1,\infty,-\infty}(\Omega;\mathbb{R}^3) = \{y; y \text{ homeomorphism, } y \in W^{1,\infty}(\Omega;\mathbb{R}^3),\\ y^{-1} \in W^{1,\infty}(y(\Omega);\Omega) \text{ and } \det \nabla y > 0 \text{ a.e. on } \Omega\ \}$$

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Notice that

- In this case the set of deformations *has* a group structure
- There actually ex. $\gamma > 0$ s.t. $\det \nabla y \geq \gamma$ a.e. on Ω $\rightsquigarrow$ this implies that condition (2) *does not pose any restriction*.

Posing the problem

$$\left.\begin{array}{ll}\text{Minimize} & \int_\Omega W(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in \mathcal{A}.\end{array}\right\} \qquad (1)$$

with

$$\mathcal{A} = W^{1,\infty,-\infty}(\Omega; \mathbb{R}^3)$$

Under which (minimal) conditions on the stored energy W does (1) admit a solution?

Refining the problem

- A usual approach is to employ the *direct method*
- Crucial ingredients:
  1. closedness of $\mathcal{A}$ under appropriate weak convergence ($\rightsquigarrow$ this is OK in our case under the convergence below)
  2. coercivity of W that enforces this weak convergence of the minimization sequence
  3. A corresponding lower semicontinuity of $\int_\Omega W$

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If $\{y_k\}_{k>0}$ is a bounded sequence of bi-Lipschitz, orientation preserving homeomorphisms such that

$$y_k \overset{*}{\rightharpoonup} y$$

under which minimal conditions on W does it hold that

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x?$$

Sufficient conditions

- Since $\mathcal{A} \subset W^{1,\infty}(\Omega;\mathbb{R}^3)$, W defines a weakly lower semicontinuous functional if it is quasiconvex
- Yet, the condition

$$W(Y) \leq \frac{1}{|\Omega|}\int_\Omega W(\nabla\varphi)\,\mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times3}$ and all $\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^3); \varphi = Yx$ on $\partial\Omega$. is not natural.

- *Why should we test also with non-deformations* ?

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- *Why should we test also with non-deformations* ?

Necessary and sufficient conditions

- Remeber, the condition

$$W(Y) \leq \inf_{\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^d); \varphi = Yx \text{ on } \partial\Omega} \frac{1}{|\Omega|} \int_\Omega W(\nabla \varphi)\,\mathrm{d}x.$$

  is not natural.
- *Why should we test also with non-deformations* $\rightsquigarrow$ particularly when looking at the principle of virtual displacements?

Conjecture

If $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms such that $y_k \overset{*}{\rightharpoonup} y$ then

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x$$

if and only if *it is bi-quasiconvex*, i.e.

$$W(Y) \leq \frac{1}{|\Omega|} \int_\Omega W(\nabla \varphi)\,\mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times 3}$ with $\det Y > 0$ and all $\varphi \in \mathcal{A}; \varphi = Yx$ on $\partial\Omega$.

Necessary and sufficient conditions

- this is still an open problem ... but ...

Necessary and sufficient conditions

Proposition [B.B& M.Kr., 2013]

If $\{y_k\}_{k>0}$ is a bounded sequence of bi-Lipschitz, orientation preserving homeomorphisms in the plane (i.e. $\Omega \subset \mathbb{R}^2, y_k : \Omega \mapsto \mathbb{R}^2$) such that $y_k \overset{*}{\rightharpoonup} y$ then

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x$$

if and only if

$$W(Y) \leq \frac{1}{|\Omega|} \int_\Omega W(\nabla \varphi)\, \mathrm{d}x.$$

for all $Y \in \mathbb{R}^{2\times 2}$ with $\det Y > 0$ and all $\varphi \in \mathcal{A}; \varphi = Yx$

Notions of quasiconvexity

The idea that one should verify the Jensen inequality only for functions that are "deformations" appears also in:

- $W^{1,p}$-quasiconvexity
- Orientation-preserving quasiconvexity

[Ball, Murat; 1988], [Koumatos, Rindler, Wiedemann; 2014]

Constructing a cut-off as key ingredient

- The key ingredient in the proof of this proposition is the construction of some kind of cut-off
- Indeed, take $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} Yx$ (for simplicity)
- If $\forall k$ we had $y_k = Yx$ on $\partial\Omega$, then (from def.)

$$W(Y) \leq \liminf_{k\to\infty} \frac{1}{|\Omega|} \int_\Omega W(\nabla y_k)\,\mathrm{d}x.$$

$\Rightarrow$ weak* lower semicontinuity

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Constructing a cut-off as key ingredient

Reformulating once again

Suppose $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} Yx$. Then find another sequence $\{w_k\}_{k>0}$ of bi-Lipschitz, orientation preserving homeomorphisms such that

- $w_k = Yx$ on $\partial\Omega$,
- $\big|\{w_k \neq y_k\}\big| \to 0$.

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- $w_k = Yx$ on $\partial\Omega$,
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- this is a consequence of using *Young measures* $\rightsquigarrow$ a useful tool in such situations
- Notice: $\det Y > 0$

[Kinderlehrer, Pedregal; 1991, 1992, 1994]

Constructing a cut-off as key ingredient

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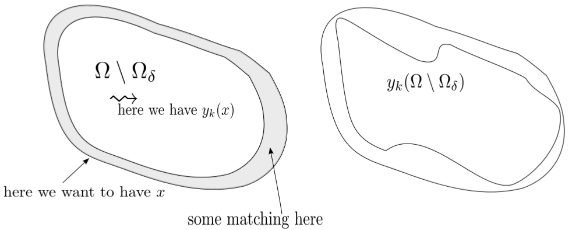
We can imagine a cut-off by the following picture ($Y = \mathrm{Id}$ here):



Cut-off and surjectivity of the trace operator

Notice: the cut-off technique is very much related to characterizing the trace operator.

- What we need to do is to find some $w_k \in W^{1,p,-p}(\Omega)$ on Ω_δ with *prescribed* boundary data, such that the norm of w_k is controlled by a "suitable" norm at the boundary
- This is is equivalent to constructing an extension operator; in other words

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We seek a characterization of the set $\mathcal{X}^p$ such that

$$\mathrm{Tr} : W^{1,p,-p}(\Omega; \mathbb{R}^2) \xrightarrow{onto} \mathcal{X}^p$$

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We seek a characterization of the set $\mathcal{X}^p$ such that

$$\mathrm{Tr} : W^{1,p,-p}(\Omega;\mathbb{R}^2) \xrightarrow{onto} \mathcal{X}^p$$

$\rightsquigarrow$ this is completely open unless $p = \infty$ and $n = 2$.

Sidenote: Characterizing the trace operator is of independent interest

For which $g : \partial\Omega \to \mathbb{R}^n$ there is $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ such that $\det \nabla y > 0$ and $y = g$ on $\partial\Omega$?

Constructing a cut-off - difficulties

- How is the cut-off constructed usually?
- Take η_ℓ smooth cut-off function and take the convex combination

$$\eta_\ell y_k + (1 - \eta_\ell) Yx$$

- But our constrains $\det > 0$ as well as the invertibility *are not convex*

  $\rightsquigarrow$ we may easily "fall out" from the set of deformations

  .... *but* ....

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Cut-off in the plane

[Benešová & M.K.; 2013]

Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Let $\{y_k\}_{k>0}$ be a bounded sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} y$. Then there exists its (not relabeled) subsequence and another bounded sequence $\{w_k\}_{k>0}$ of bi-Lipschitz, orientation preserving homeomorphisms such that

- $w_k = y$ on $\partial\Omega$,
- $\big|\{w_k \neq y_k\}\big| \to 0$,
- $\liminf \int_\Omega W(\nabla y_k)\mathrm{d}x = \liminf \int_\Omega W(\nabla w_k)\mathrm{d}x$.

Working in the plane

- Also working in the plane makes the situation simpler
- Here, we rely on two crucial things:
    1. The boundary of domains in the plane is *one-dimensional* (e.g. the boundary of the some square)
    2. In the plane, we have bi-Lipschitz *extension* theorems at our disposal

Bi-Lipschitz extension in the plane

[Daneri, Pratelli; 2011]

There exists a geometric constant C such that every L bi-Lipschitz map u defined on the boundary of the unit square admits a CL^4 bi-Lipschitz extension into the square that coincides with u on the boundary.

[Tukia; 1980], [Huuskonen,Partanen,Väisälä; 1995], [Tukia,Väisälä;1981,1984]

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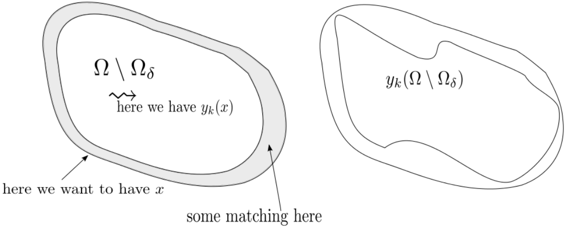
Bi-Lipschitz extension in the plane

Why is this useful in our case?

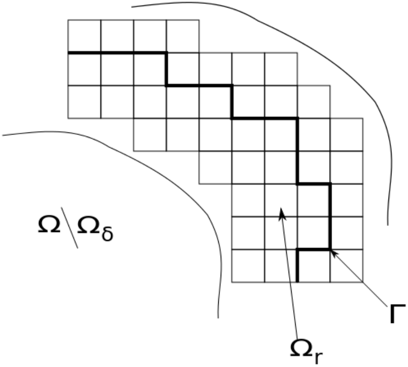
- Remember, we wanted to solve a boundary value problem
  $\rightsquigarrow$ *now we can, but on the boundary of the square....*
- So we introduce squares in the grey area



Bi-Lipschitz extension in the plane

Why is this useful in our case?

- Remember, we wanted to solve a boundary value problem
  $\rightsquigarrow$ *now we can, but on the boundary of the square....*
- So we introduce squares in the grey area



Proof-Summary

- We reduced the problem of constructing a cut-off to constructing it just on the crosses
- $\rightsquigarrow$ thus, we can define a "matching function" that is still bi-Lipschitz on the grid of the squares
- The bi-Lipschitz extension theorem then allows to get a homeomorphism inside
- Recall: this allows us to have the quasiconvexity condition (*principle of virtual displacements*) tested just by deformations (in order to anyway obtain weak* lower semicontinuity)

Summary

- It is relevant in elasticity to solve minimization problems on subsets of Sobolev functions, where *non-linear, non-convex* restrictions are posed
- Although sufficiency conditions for existence of minima are generally known, if and only if conditions are still a challenge
- Here we extended the quasiconvexity condition also to the case when minimizing over bi-Lipschitz, orientation preserving functions
- A larger class of stored energy functions can be now admitted

Thank you for your attention!

B. Benešová, M.K.: Characterization of gradient Young measures generated by homeomorphisms in the plane. *ESAIM COCV*
http://dx.doi.org/10.1051/cocv/2015003