Duality-based model adaptivity in context of multiscale methods

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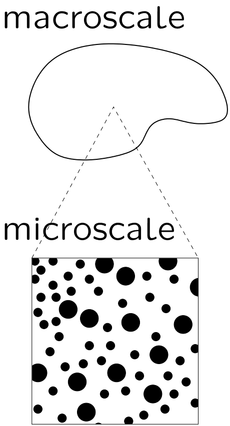
An overview of multiscale methods

What are multiscale problems?

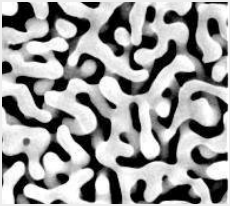
- A modeling problem is of multiscale character if relevant physical processes act on highly different length scales.
- Hard to simulate due to very high computational costs



What are multiscale problems?

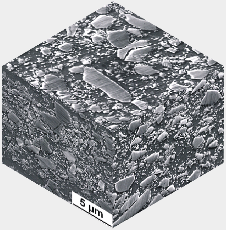
Flow in porous media

- Macroscopic flow behaviour
- Microscopic dependence of the permeability



Heterogeneous materials in structural mechanics

- Macroscopic, mechanical stress tensor
- Elastic/inelastic behaviour is highly dependent on the microstructure



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Fraunhofer ITWM, Kaiserslautern

A model problem

Model problem

Find $u^\varepsilon \in H_0^1(\Omega)$ s. t.

$$a^\varepsilon(u^\varepsilon, \varphi) = \left(A^\varepsilon(x)\nabla u^\varepsilon , \nabla\varphi\right) = (f, \varphi) \quad \forall\varphi \in H_0^1(\Omega), \qquad (\mathrm{I})$$

where A^ε contains *multiscale features*.

Assumptions

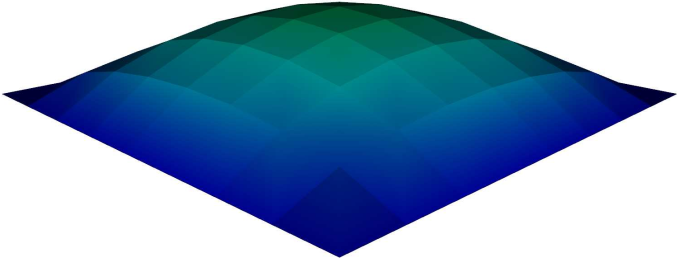
$$A^\varepsilon \in L^\infty(\Omega)^{d\times d},$$

$$A_{ij}^\varepsilon(x) = A_{ji}^\varepsilon(x) \quad \forall i, j \text{ a. e. on } \Omega,$$

$$\alpha|\xi|^2 \le (\xi, A^\varepsilon(x)\xi) \le \beta|\xi|^2 \quad \forall\xi \in \mathbb{R}^d \text{ a. e. on } \Omega.$$

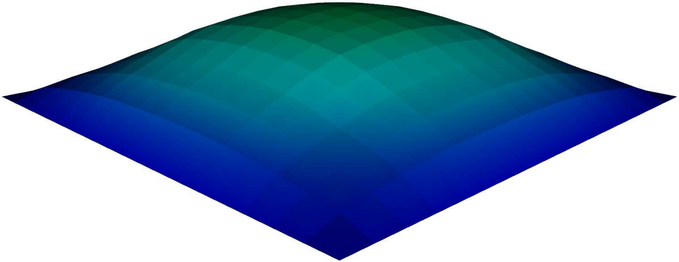
A model problem

81 degrees of freedom:



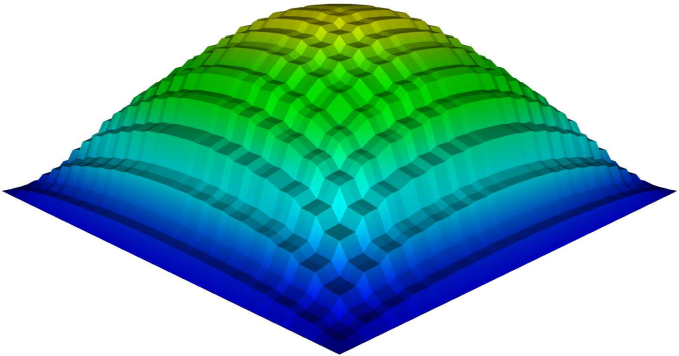
A model problem

289 degrees of freedom:



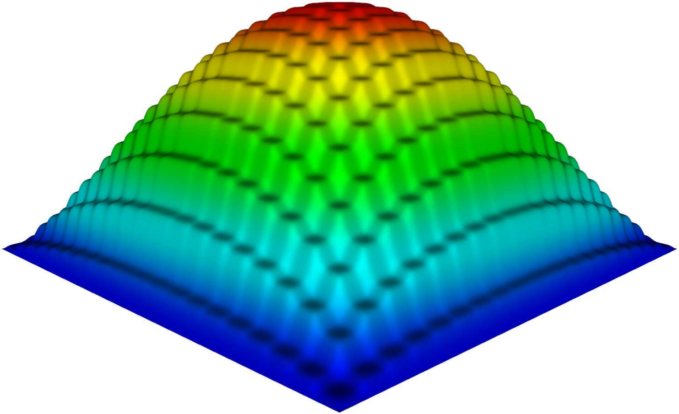
A model problem

4 225 degrees of freedom:



A model problem

263 169 degrees of freedom:



An (incomplete) overview

Multiscale modeling strategies

- Use structure of the ansatz-spaces

$$a^{\varepsilon}(u, \varphi) = (f, \varphi) \quad \forall \varphi \in V, \;\; V = V_c \oplus V_f .$$

→ Variational Multiscale Methods
(Hughes et al. '98, Brezzi '99)

- Use structure of the operator

$$A^{\epsilon}(x) = A\Big(x, \frac{x}{\varepsilon}\Big)$$

→ Homogenisation theory (e.g., Allaire '92)
→ Heterogeneous Multiscale Methods (E, Engquist '03)

- Physical upscaling principles

→ Computational Homogenization
(Hill '63, Geers et al. '10)
→ Averaging principles (e.g., geometric mean)

Heterogeneous Multiscale Methods

Heterogeneous Multiscale Methods

$$A^{\epsilon}(x) = A\left(x, \frac{x}{\varepsilon}\right)$$

References

- *"The heterogeneous multiscale method"*, E, Engquist '03
- *"Homogenization and Two-Scale Convergence"*, Allaire '92
- *"A posteriori error estimates for the heterogeneous multiscale finite element method for elliptic homogenization problems"*, Ohlberger '05

Heterogeneous Multiscale Methods

Model problem

$$\big(A^\varepsilon(x)\nabla u^\varepsilon\,,\nabla\varphi\big) = (f\,,\varphi) \quad \forall\varphi\in H^1_0(\Omega). \tag{I}$$

- Assume a scale separation assumption to hold true:

$$A^\varepsilon(x) = A\big(x, \tfrac{x}{\varepsilon}\big),\ \text{with } A\in C^{0,1}\big(\Omega, L^\infty_{\text{per}}(Y)^{d\times d}\big).$$

Homogenized problem

$$\big(A^0(x)\nabla u^0\,,\nabla\varphi\big) = (f\,,\varphi) \quad \forall\varphi\in H^1_0(\Omega),\ \text{where}$$

$$A^0_{ij}(x) = \int_Y A(x,y)\left[\mathbf{e}_i + \nabla_y\omega_i(x,y)\right]\cdot\left[\mathbf{e}_j + \nabla_y\omega_j(x,y)\right]\mathrm{d}y$$

and $\omega_i\in\tilde H^1_{\text{per}}(Y)$ are solutions of cell problems

$$\int_Y A(x,y)\left[\mathbf{e}_i + \nabla_y\omega_i(x,y)\right]\cdot\nabla\varphi\,\mathrm{d}y \quad \forall\varphi\in\tilde H^1_{\text{per}}(\Omega).$$

Heterogeneous Multiscale Methods

Model problem

$$\big(A^\varepsilon(x)\nabla u^\varepsilon\,,\nabla\varphi\big) = (f,\varphi) \quad \forall\varphi \in H^1_0(\Omega). \qquad \text{(I)}$$

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$$\int_Y A(x,y)\left[\boldsymbol{e}_i + \nabla_y\omega_i(x,y)\right]\cdot\nabla\varphi\,\mathrm{d}y \quad \forall\varphi \in \tilde{H}^1_{\text{per}}(\Omega).$$

Heterogeneous Multiscale Methods

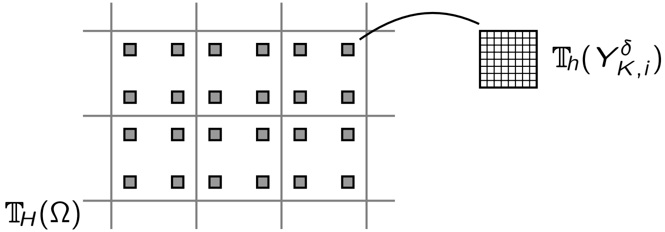
Reformulation (Allaire, Ohlberger)

$$\int_\Omega \fint_{Y^\delta} A(x, \tfrac{y}{\delta}) \left[\nabla_x u^0 + \nabla_y \mathcal{R}(u^0)(x,y)\right] \cdot \left[\nabla_x \varphi^0 + \nabla_y \mathcal{R}(\varphi^0)(x,y)\right] \mathrm{d}y \mathrm{d}x = \int_\Omega f \varphi^0 \mathrm{d}x,$$

with $\mathcal{R}(\varphi)(x,y)$ defined as the solution of

$$\int_{Y^\delta} A(x, \tfrac{y}{\delta}) \left[\nabla_x \varphi(x) + \nabla_Y \mathcal{R}(\varphi)(x,y)\right] \cdot \nabla \psi \mathrm{d}y = 0 \quad \forall \psi.$$

Heterogeneous Multiscale Methods



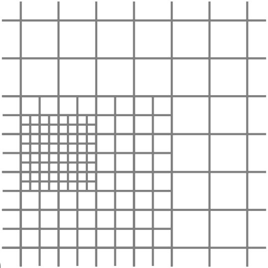
Heterogeneous Multiscale Formulation

$$\sum_{K\in\mathbb{T}_H} |K| \fint_{Y^\delta_{Ki}} A^\varepsilon(x)\,\nabla\big[U + \mathcal{R}^h_{Ki}(U)(x)\big]\cdot\nabla\big[\varphi + \mathcal{R}^h_{Ki}(\varphi)(x)\big]\mathrm{d}x$$

$$= \sum_{K\in\mathbb{T}_H} |K| \sum_i q_i f(x_i)\varphi(x_i) \quad \forall\varphi\in V^H(\Omega)$$

Multiscale framework

Multiscale framework

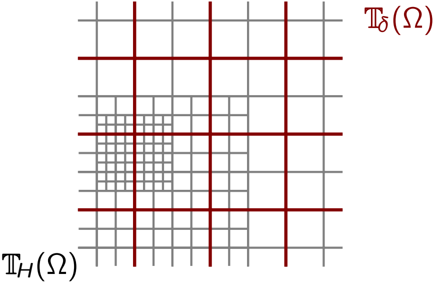


$\mathbb{T}_H(\Omega)$

References

- *"Duality-based adaptivity in finite element discretization of heterogeneous multiscale problems"*
  M., Rannacher, submitted '14

Multiscale framework



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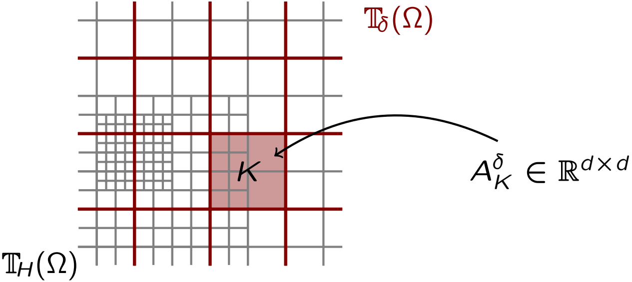
Multiscale framework



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Multiscale framework

Numerically homogenized, continuous

Given a model $A^\delta : \mathbb{T}_\delta \to \mathbb{R}^{d\times d}$, find $u^\delta \in H^1_0(\Omega)$ s. t.

$$\big(A^{\delta}(x)\nabla u^{\delta}, \nabla\varphi\big) = (f, \varphi) \quad \forall\varphi \in H^1_0(\Omega). \qquad (\mathrm{II})$$

Multiscale framework

Numerically homogenized, continuous

Given a model $A^\delta : \mathbb{T}_\delta \to \mathbb{R}^{d\times d}$, find $u^\delta \in H^1_0(\Omega)$ s. t.

$$\big(A^{\delta}(x)\nabla u^{\delta}\,,\nabla\varphi\big) = (f,\varphi) \quad \forall \varphi \in H^1_0(\Omega). \qquad (\mathrm{II})$$

- Simple averaging:

$$A^\delta_{ij}\Big|_K = \fint_{Y^\delta_K} A^\varepsilon(y)\,\mathrm{d}y, \qquad A^\delta_{ij}\Big|_K = \exp\Big(\fint_{Y^\delta_K} \log(A^\varepsilon(y))\,\mathrm{d}y\Big).$$

Multiscale framework

Numerically homogenized, continuous

Given a model $A^\delta : \mathbb{T}_\delta \to \mathbb{R}^{d\times d}$, find $u^\delta \in H^1_0(\Omega)$ s. t.

$$\left(A^\delta(x)\nabla u^\delta, \nabla\varphi\right) = (f, \varphi) \quad \forall\varphi \in H^1_0(\Omega). \tag{II}$$

- Mathematical homogenization:

$$A^\delta_{ij}\Big|_K = \fint_{Y^\delta_K} A^\varepsilon(x)\big(\boldsymbol{e}_i + \nabla w_i(x)\big)\cdot\big(\boldsymbol{e}_j + \nabla w_j(x)\big)\mathrm{d}x,$$

with $w_i \in H^1_{\mathrm{per}}(Y^\delta_K)$ solving a cell/sampling problem:

$$\fint_{Y^\delta_K} A^\varepsilon(x)\,(\boldsymbol{e}_i + \nabla w_i(x))\cdot\nabla\varphi\,\mathrm{d}x = 0 \quad \forall\,\varphi \in H^1_{\mathrm{per}}(Y^\delta_K).$$

Multiscale framework

Numerically homogenized, semi discretized

Find $u^{\delta,h} \in H^1_0(\Omega)$ s. t.

$$\left(A^{\delta,h}(x)\nabla u^{\delta,h}, \nabla\varphi\right) = (f, \varphi) \quad \forall\varphi \in H^1_0(\Omega), \qquad \text{(III)}$$

where $A^{\delta,h} : \Omega \to \mathbb{R}^{d\times d}$ is an approximation on A^δ.

Multiscale framework

Numerically homogenized, semi discretized

Find $u^{\delta,h} \in H_0^1(\Omega)$ s. t.

$$\left(A^{\delta,h}(x)\nabla u^{\delta,h}, \nabla\varphi\right) = (f,\varphi) \quad \forall\varphi \in H_0^1(\Omega), \qquad (\,\text{III}\,)$$

where $A^{\delta,h} : \Omega \to \mathbb{R}^{d\times d}$ is an approximation on A^δ.

- Simple averaging:

$$A_{ij}^{\delta,h}\Big|_K = Q^h_{Y_K^\delta}\left(A^\varepsilon\right), \qquad \ldots$$

Multiscale framework

Numerically homogenized, semi discretized

Find $u^{\delta,h} \in H^1_0(\Omega)$ s. t.

$$\Big(A^{\delta,h}(x)\nabla u^{\delta,h}, \nabla\varphi\Big) = (f,\varphi) \quad \forall\varphi \in H^1_0(\Omega), \qquad \text{(III)}$$

where $A^{\delta,h} : \Omega \to \mathbb{R}^{d\times d}$ is an approximation on A^δ.

- Homogenization: Discretize with meshes $\mathbb{T}_h(Y^\delta_K)$:

$$A^{\delta,h}_{ij}\Big|_K = \fint_{Y^\delta_K} A^\varepsilon(x)\big(\mathbf{e}_i + \nabla w^h_i(x)\big)\cdot\big(\mathbf{e}_j + \nabla w^h_j(x)\big)\mathrm{d}x,$$

with $w^h_i \in V^h(Y^\delta_K)$ solving a cell/sampling problem:

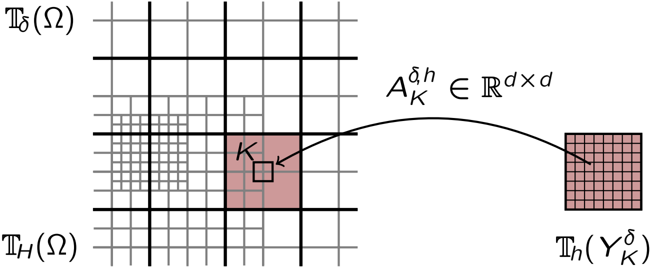
$$\fint_K A^\varepsilon(x)\big(\mathbf{e}_i + \nabla w^h_i(x)\big)\cdot\nabla\varphi\,\mathrm{d}x = 0 \quad \forall\,\varphi \in V^h(K).$$

Multiscale framework

Numerically homogenized, fully discretized

Find $U \in V^H(\Omega)$ s. t.

$$\left(A^{\delta,h}(x)\nabla U\,,\nabla\varphi\right) = (f\,,\varphi) \quad \forall\varphi \in V^h(\Omega). \qquad \text{(IV)}$$



Model: $A^\delta : \mathbb{T}_\delta \to \mathbb{R}^{d\times d}$ Discretization: $\mathbb{T}_H$, $\left\{\mathbb{T}_h(Y_K^\delta)\right\}$

Multiscale framework

Dual problem

Let $j : H_0^1(\Omega) \to \mathbb{R}$ be a functional; define a dual problem:
Find $z^\varepsilon \in H_0^1(\Omega)$ s. t.

$$(A^\varepsilon(x)\nabla\varphi\,, \nabla z^\varepsilon) = \langle j, \varphi\rangle \quad \forall\varphi \in H_0^1(\Omega). \qquad (\mathrm{V})$$

This leads to the error identity:

$$\langle j, e\rangle = \langle j, u^\varepsilon\rangle - \langle j, U\rangle = \underbrace{(f, z^\varepsilon) - \left(A^{\delta,h}\nabla U, \nabla z^\varepsilon\right)}_{\mathrm{I}}$$

$$+ \underbrace{\left((A^{\delta,h} - A^\delta)\nabla U, \nabla z^\varepsilon\right)}_{\mathrm{II}} + \underbrace{\left((A^\delta - A^\varepsilon)\nabla U, \nabla z^\varepsilon\right)}_{\mathrm{III}}.$$

Multiscale framework

Residual on the macroscale:

$$(\mathrm{I}) = \sum_{K\in\Pi_H} \left(f, z^\varepsilon - \psi^H\right)_K - \left(A^{\delta,h}\nabla U, \nabla z^\varepsilon - \nabla\psi^H\right)_K$$

Residual on the microscale:

$$(\mathrm{II}) = \sum_{K\in\Pi_\delta} \left((A^{\delta,h} - A^\delta)\nabla U, \nabla z^\varepsilon\right)_K$$

In case of homogenization:

$$(\mathrm{II}) = \sum_{K\in\Pi_\delta} \sum_{Q\in\Pi_h(Y_K^\delta)} \sum_{i,j} \left(\fint_K \partial_i U \partial_j z^\varepsilon \,\mathrm{d}x\right) \left\{-\int_Q A^\varepsilon(\boldsymbol{e}_j + \nabla w_j^h)\cdot(\nabla w_i - \nabla\psi_{w_i}^h)\mathrm{d}y\right\}$$

Multiscale framework

Remark: Approximating the dual solution

- It is computationally infeasible to resolve z^ε completely.

- It is crucial to get a good (local) approximation of z^ε for a *quantitative* estimate of the *model error*
  - (I), (II) correspond to *moments of first order*:

$$(\mathrm{II}) = \sum_{K \in \Pi_\delta} \Big((A^{\delta,h} - A^\delta) \nabla U, \underbrace{\nabla z^\varepsilon}_{\text{fluct.}} \Big)_K$$

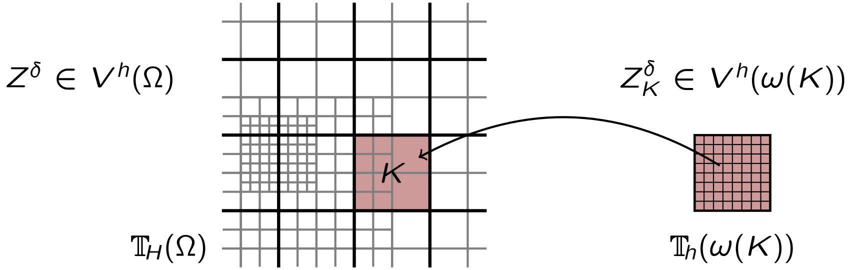
  - (III) corresponds to a *moment of second order*:

$$(\mathrm{III}) = \sum_{K \in \Pi_\delta} \Big(\underbrace{(A^\delta - A^\varepsilon)}_{\text{fluct.}} \nabla U, \underbrace{\nabla z^\varepsilon}_{\text{fluct.}} \Big)_K$$

Multiscale framework

Remark: Approximating the dual solution

- Solve a reduced dual problem: Find $Z^\delta \in V^H(\Omega)$ s. t.
$$\left(A^\delta(x)\nabla\varphi\,,\nabla Z^\delta\right) = \langle j,\varphi\rangle \quad \forall\varphi \in V^H(\Omega).$$
- Additional local enhancement $Z^\delta + Z^\delta_K$ on K:
$$\left(A^\epsilon(x)\nabla\varphi\,,\nabla(Z^\delta + Z^\delta_K)\right)_{\omega(K)} = \langle j,\varphi\rangle \;\forall\varphi \in V^h(\omega(K)).$$



Model adaptation

Model adaptation

Model and model error indicator

$$\text{Model: } A^\delta : \mathbb{T}_\delta \to \mathbb{R}^{d\times d},$$

$$\theta^\delta = \sum_{K\in\Pi_\delta} \eta_K^\delta, \quad \eta_K^\delta = \big((A^\delta - A^\varepsilon)\nabla U, \nabla z^\varepsilon\big)_K .$$

Model adaptation strategies:

- Adapt $\{Y_K^\delta : K \in \mathbb{T}_\delta(\Omega)\}$, $\mathbb{T}_\delta(\Omega)$
- Adapt A^δ (for fixed $\mathbb{T}_\delta(\Omega)$)
  $\longrightarrow$ switch model
  $\longrightarrow$ model optimization

Model adaptation

Strategy I (M., Rannacher '14)

- use simple averaging strategy (e. g. geometric average)
- adapt $\mathbb{T}_\delta(\Omega)$ locally with the error indicator η_K^δ.

- Adapt $\mathbb{T}_H(\Omega)$ and $\{h_K : K \in \mathbb{T}_\delta(\Omega)\}$ with η_K^H, or η_K^h.
- If η_K^δ is large for some $K \in \mathbb{T}_\delta(\Omega)$
  - split K into 2^d sampling regions K_i,
  - shrink sampling region Y_K^δ by a factor of 2^{-d}, set $h_{K_i} = h_K$
- This also only requires a reduced dual problem z^δ.

Model adaptation

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- This also only requires a reduced dual problem z^δ.

Model adaptation

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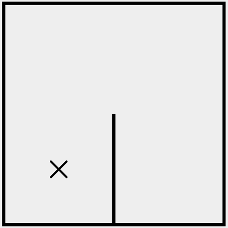
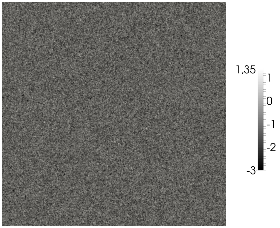
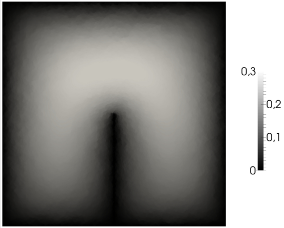
Model adaptation

Numerical example







Microstructure (a log-normally distributed random field with Gaussian correlation):

$$A^{\varepsilon}(x) = I_d \times \gamma \times \exp(10 \times g(x)/255)$$

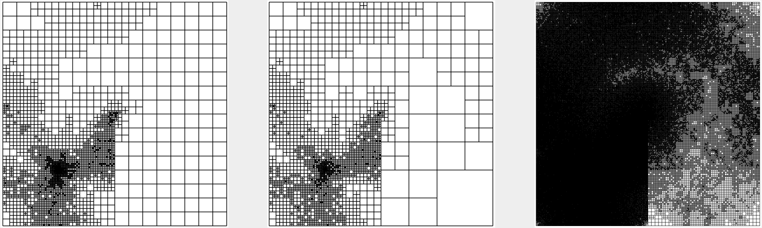
Quantity of interest:

$$\langle j, \varphi \rangle := \partial_{x_2} \varphi(\hat{x}), \qquad \hat{x} = (0.25,\ 0.25).$$

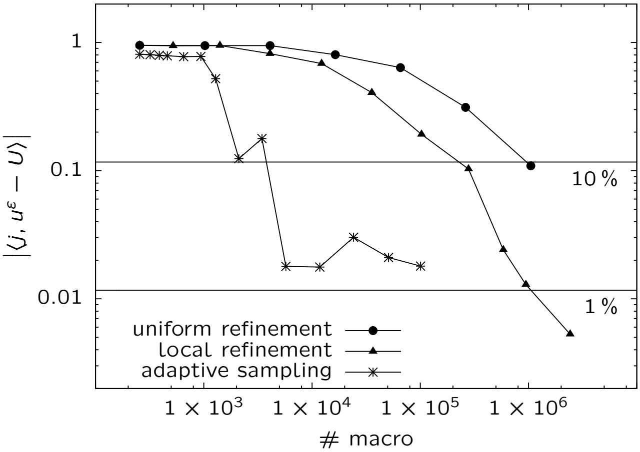
Model adaptation

strategy	L^2 error	$\lvert\langle j, u^\varepsilon - U\rangle\rvert$
arithmetic	6.85e-2 (40%)	9.40e-1
geometric	3.66e-3 (2.2%)	7.93e-1
HMM	2.87e-3 (1.7%)	7.93e-1

(a) $\mathbb{T}_H$ (b) $\mathbb{T}_\delta$ (c) $\mathbb{T}_H$



Model adaptation



- Main mechanism for saving: Approximation property of geometric average

Model adaptation

Strategy II: Optimization problem (M. '15)

For a given Π_δ and initial $A^{\delta,0}$ define an *optimal model*,

$$\tilde{A}^\delta \ : \ \Pi_\delta \to \mathbb{R}^{d\times d},$$

to be a solution of

$$\underset{A^\delta : \Pi_\delta \to \mathbb{R}^{d\times d}}{\operatorname{argmin}} \sum_{K\in\Pi_\delta} \Big(\underbrace{(A^\delta - A^\varepsilon)\nabla U(A^\delta), \nabla z^\varepsilon}_{=\eta_K}\Big)_K^2 + \kappa |A_K^\delta - A_K^{\delta,0}|^2$$

subject to

$$(\xi A^\delta(x), \xi) \geq \alpha^* |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \forall x \in \Omega,$$

$$\big(A^\delta \nabla U(A^\delta), \nabla\varphi\big) = \big(f, \varphi\big) \quad \forall \varphi \in V_H.$$

Model adaptation

Algorithm (pure model adaption)

1. Compute $A^{\delta,0}$, $A^{\delta} \leftarrow A^{\delta,0}$
2. Solve primal and (reduced) dual problem $U(A^{\delta})$, $Z^{\delta}(A^{\delta})$
3. Compute estimators

$$\eta_K = \left((A^{\delta} - A^{\varepsilon})\nabla U(A^{\delta}), \nabla Z^{\delta} + \nabla Z^{\delta}_K\right)$$

   and solve

$$\left(JJ^T + \lambda I\right)\delta A^{\delta} = -J^T\eta.$$

4. $A^{\delta} \leftarrow A^{\delta} + \delta A^{\delta}$
5. If stopping criterion not reached continue at (2)

Numerical results

Example: Advection-diffusion problem

$$\gamma(\nabla u^\varepsilon, \nabla\varphi) + (\boldsymbol{b}^\varepsilon \cdot \nabla u^\varepsilon, \varphi) = (f, \varphi),$$

with dominant, microscopic transport $\boldsymbol{b}^\varepsilon$.

Effective Model

$$(A^\delta \nabla u^\delta, \nabla\varphi) + (\boldsymbol{b}^\delta \cdot \nabla u^\delta, \varphi) = (f, \varphi),$$

with $\boldsymbol{b}^\delta(K) := \fint_K \boldsymbol{b}^\varepsilon \, \mathrm{d}x$.

Modified model-error indicator

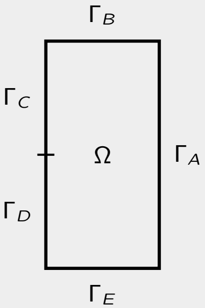
$$\eta_K^\delta = \big((\gamma\,\mathrm{Id} - A^\delta)\nabla u^\delta, \nabla z^\varepsilon\big)_K - \big((\boldsymbol{b}^\varepsilon - \boldsymbol{b}^\delta) \cdot \nabla u^\delta, z^\varepsilon\big)_K.$$

Numerical results

Quantity of interest:

$$\langle j, \varphi \rangle = \int_{\Gamma_B} \varphi \, \mathrm{d}o_x$$

Domain:



Two different choices for adv. field $\boldsymbol{b}^{\varepsilon}$:

- Periodic coefficient:

$$\boldsymbol{b}^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = (-1)^{\lfloor x/\varepsilon \rfloor + \lfloor y/\varepsilon \rfloor} \, \boldsymbol{b}\big(\hat{\boldsymbol{x}}.\big)$$

- Random coefficient:

$$\boldsymbol{b}^{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \operatorname{curl} \Psi^{\varepsilon},$$

with random scalar $\Psi^{\varepsilon} \in C^2(\Omega)$.

Numerical results

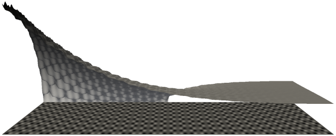
	$L^2(\Omega)$	$\lvert\langle j, U\rangle\rvert$	$\lvert\langle j, u^\varepsilon - U\rangle\rvert$	$\lvert\tilde{\theta}^\delta\rvert$	I_{eff}	I_{loc}
1	5.77	4.31	-2.6e+0 (142 %)	-2.6e+0	1.01	1.67
7	1.34	1.79	-1.02e-2 (0.57 %)	-1.77e-2	1.61	7.86
15	1.33	1.78	5.40e-3 (0.31 %)	-2.00e-3	-0.14	21.81

(a) Fully resolved dual solution, periodic coefficients

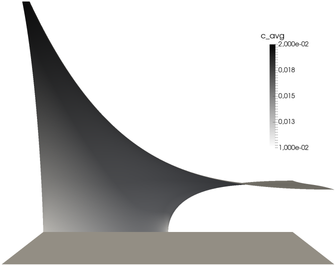
	$L^2(\Omega)$	$\lvert\langle j, U\rangle\rvert$	$\lvert\langle j, u^\varepsilon - U\rangle\rvert$	$\lvert\tilde{\theta}^\delta\rvert$	I_{eff}	I_{loc}
1	5.77	4.31	-2.6e+0 (142 %)	-6.2e+0	2.44	1.66
6	1.33	1.76	2.31e-2 (1.30 %)	-3.37e-2	-1.46	5.04
15	1.32	1.74	4.34e-2 (2.44 %)	-3.31e-3	-0.08	10.1

(b) Reduced, locally enhanced dual solution, periodic coefficients

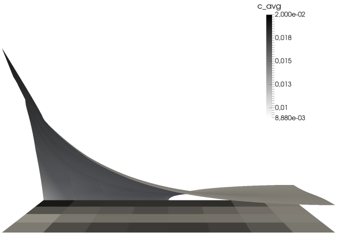
Numerical results



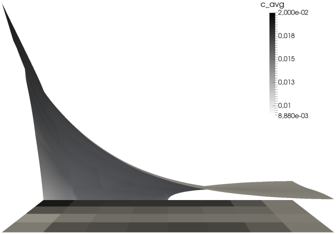
(a) Reference solution



(b) Initial solution



(c) Fully resolved dual solution



(d) Locally enhanced dual solution

Numerical results

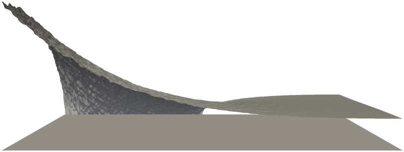
	$L^2(\Omega)$	$\lvert\langle j, U\rangle\rvert$	$\lvert\langle j, u^\varepsilon - U\rangle\rvert$	$\lvert\tilde{\theta}^\delta\rvert$	I_{eff}	I_{loc}
1	4.43e-1	3.86e-1	-1.69e-1 (77.9 %)	-1.69e-1	1.00	2.21
11	1.39e-1	2.26e-1	-9.14e-3 (4.21 %)	-9.35e-3	1.00	13.2
15	1.13e-1	2.23e-1	-6.28e-3 (2.89 %)	-6.51e-3	0.99	17.3

(c) Fully resolved dual solution, random coefficients

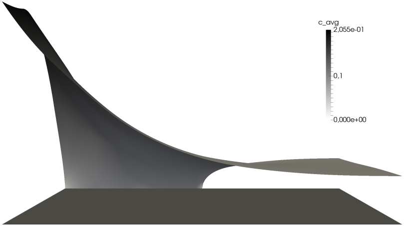
	$L^2(\Omega)$	$\lvert\langle j, U\rangle\rvert$	$\lvert\langle j, u^\varepsilon - U\rangle\rvert$	$\lvert\tilde{\theta}^\delta\rvert$	I_{eff}	I_{loc}
1	4.43e-1	3.86e-1	-1.69e-1 (77.9 %)	-2.96e-1	1.76	2.15
10	7.99e-2	2.20e-1	-3.02e-3 (1.39 %)	-1.27e-2	4.18	4.91
15	9.70e-2	2.14e-1	3.07e-3 (1.41 %)	7.00e-3	2.28	10.4

(d) Reduced, locally enhanced dual solution, random coefficients

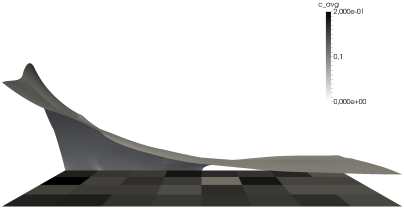
Numerical results



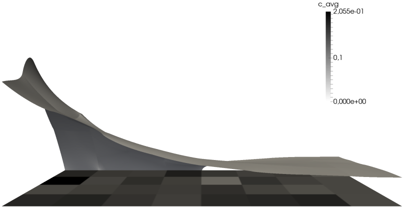
(e) Reference solution



(f) Initial solution



(g) Fully resolved dual solution



(h) Locally enhanced dual solution

Conclusion

Conclusion

Conclusion

- Multiscale formulations are based on structure in ansatz spaces (VMM) or in the operator (HMM).
- Error estimation with Dual-Weighted Residual method.
- A modified heterogeneous multiscale scheme was presented, suitable for error splitting.
- Adjustment of an effective model $A^\delta : \Pi_\delta \to \mathbb{R}^{d\times d}$ by
  - Adaptation of discretization $\left\{ Y_K^\delta : K \in \mathbb{T}_\delta(\Omega) \right\}$, $\mathbb{T}_\delta(\Omega)$
  - Model adaptation A^δ (for fixed $\mathbb{T}_\delta(\Omega)$)

Thank you for your attention!

Special thanks to:

Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences

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