On the analysis of unsteady flows of implicitly constituted incompressible fluids

Miroslav Bulíček

Mathematical Institute of the Charles University
Sokolovská 83, 186 75 Prague 8, Czech Republic

P. Gwiazda, **J. Málek** and **A. Świerczewska-Gwiazda**

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Balance equations

We consider flow of a homogeneous incompressible fluid under constant temperature

$$\boxed{\operatorname{div} v = 0}$$

$$\boxed{v_{,t} + \operatorname{div}(v \otimes v) - \operatorname{div} \mathbf{S} = -\nabla p + f}$$

$$\boxed{\mathbf{S} = \mathbf{S}^T}$$

- v is the velocity of the fluid
- p is the pressure
- f external body forces ($\equiv \mathbf{0}$)
- $\mathbf{S}$ is the constitutively determined part of the Cauchy stress

The Cauchy stress is given as $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$

Point-wisely given constitutive equations

- We denote by $\mathbf{D}(v)$ the symmetric part of the velocity gradient, i.e., $2\mathbf{D}(v) := \nabla v + (\nabla v)^T$.
- We assume for simplicity only point-wise relation between $\mathbf{D}$ and $\mathbf{S}$.
- We add to balance equations some implicit (constitutive) formula:

$$\boxed{\mathbf{F}(\mathbf{S}, \mathbf{D}, p, x, t, \text{temperature, concentration, etc.}) = \mathbf{0}}.$$

- In what follows we consider only:

$$\boxed{\mathbf{F}(\mathbf{S}, \mathbf{D}) = \mathbf{0}}$$

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Explicit constitutive equations

Nice "continuous" explicit models ($\mathbf{S} := \mathbf{S}(\mathbf{D})$)

- Newtonian fluid

$$\mathbf{S} = \nu_0 \mathbf{D}, \qquad \nu_0 > 0,$$

- Ladyzhenskaya (power-law like fluid)

$$\mathbf{S} = \nu_0(\nu_1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D}, \qquad r > 1, \qquad \nu_1 \geq 0.$$

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$$\mathbf{D} = \nu_0^* \mathbf{S}, \qquad \nu_0^* > 0,$$

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$$\mathbf{D} = \nu_0^*(\nu_1^* + |\mathbf{S}|^2)^{\frac{r^*-2}{2}}\mathbf{S}, \qquad r^* > 1, \qquad \nu_1 \geq 0.$$

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Explicit constitutive equations

"Discontinuous" explicit models

- Perfect plastic

$$|\mathbf{D}| = 0 \implies |\mathbf{S}| \le 1$$
$$|\mathbf{D}| > 0 \implies \mathbf{S} := \frac{\mathbf{D}}{|\mathbf{D}|}$$

- Bingham (Herschley-Bulkley fluid)

$$|\mathbf{D}| = 0 \implies |\mathbf{S}| \le \nu_0$$
$$|\mathbf{D}| > 0 \implies \mathbf{S} := \frac{\nu_0 \mathbf{D}}{|\mathbf{D}|} + \nu(|\mathbf{D}|)\mathbf{D}$$

- Fluids with activation criteria

$$\mathbf{S} = \nu(|\mathbf{D}|)\mathbf{D}$$

with ν being discontinuous at some d^*-the activation criterium

Implicit-like constitutive equations

Still nice continuous explicit formula

- Bingham fluid

$$\mathbf{D} = \frac{(|\mathbf{S}| - \nu_0)_+}{\nu_1|\mathbf{S}|}\mathbf{S}$$

Only fully implicit continuous choice

- Perfect plastic

$$||\mathbf{D}|\mathbf{S} - \mathbf{D}| + (|\mathbf{S}| - 1)_+ = 0$$

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Implicit formulation - maximal (monotone) graph setting

Implicit theory allows to get more models. Principle of objectivity and material isotropy imply that

- Explicit relation $\mathbf{S} = \mathbf{S}(\mathbf{D})$ - the only form

$$\mathbf{S} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{D} + \alpha_2 \mathbf{D}^2$$

  with α's dependent on invariants

- Implicit relation $\mathbf{F}(\mathbf{S}, \mathbf{D})$ - the only form

$$0 = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{D} + \alpha_2 \mathbf{D}^2 + \alpha_3 \mathbf{S} + \alpha_4 \mathbf{S}^2 + \alpha_5 (\mathbf{DS} + \mathbf{SD}) + \dots$$

  with α's dependent on invariants

Implicit formulation - maximal (monotone) graph setting

Implicit function $\mathbf{F}$ determines a graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$ (or $\mathcal{A}(t,x)$). We assume that the graph is the ψ-maximal monotone graph:

- $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$
- Monotonicity: For any $(\mathbf{S}_1,\mathbf{D}_1),(\mathbf{S}_2,\mathbf{D}_2) \in \mathcal{A}$

$$(\mathbf{S}_1-\mathbf{S}_2):(\mathbf{D}_1-\mathbf{D}_2) \geq 0$$

  No strict monotonicity is needed!
- Maximal graph: If for some $(\mathbf{S},\mathbf{D})$ there holds

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  then

$$(\mathbf{S},\mathbf{D}) \in \mathcal{A}$$

- If $\mathcal{A}$ is (t,x)-dependent some measurability w.r.t. (t,x)
- ψ and ψ^* coercivity: For any $(\mathbf{S},\mathbf{D}) \in \mathcal{A}(t,x)$

$$\mathbf{S}:\mathbf{D} \geq \alpha(\psi(\mathbf{D})+\psi^*(\mathbf{S}))-g(t,x) \tag{En}$$

  with $\alpha \in (0,1]$ and $g \in L^1$.

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What is ψ? Excursion to Orlicz setting

Assume that $\psi : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}$ is an N - function (if it depends only on the modulus then Young function), i.e.,

- ψ is convex and continuous
- $\psi(\mathbf{D}) = \psi(-\mathbf{D})$
- $$\lim_{|\mathbf{D}|\to 0_+} \frac{\psi(\mathbf{D})}{|\mathbf{D}|} = 0, \qquad \lim_{|\mathbf{D}|\to \infty} \frac{\psi(\mathbf{D})}{|\mathbf{D}|} = \infty$$

We define the conjugate function ψ^* as

$$\boxed{\psi^*(\mathbf{S}) := \max_{\mathbf{D}} (\mathbf{S} : \mathbf{D} - \psi(\mathbf{D}))}$$

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What is ψ? Excursion to Orlicz setting

- Young inequality:

$$\boxed{\mathbf{S} : \mathbf{D} \le \psi(\mathbf{D}) + \psi^*(\mathbf{S})}$$

- Orlicz spaces: The Orlicz space $L^{\psi}(\mathcal{O})^{d\times d}$ is the set of all measurable function $\mathbf{D} : \Omega \to \mathbb{R}^{d\times d}_{sym}$ such that

$$\lim_{\lambda\to\infty} \int_{\mathcal{O}} \psi(\lambda^{-1}\mathbf{D}) = 0$$

with the norm

$$\|\mathbf{D}\|_{L^{\psi}} := \inf\{\lambda;\ \int_{\mathcal{O}} \psi(\lambda^{-1}\mathbf{D}) \le 1\}$$

- Δ_2 condition

$$\psi(2\mathbf{D}) \le C_1\psi(\mathbf{D}) + C_2$$

Optimality of ψ and ψ^* - more general models

- Non-polynomial growth

$$\mathbf{S} \sim (1+|\mathbf{D}|^2)^{\frac{r-2}{2}} \ln(1+|\mathbf{D}|)\mathbf{D} \implies \psi(\mathbf{D}) \sim |\mathbf{D}|^r \ln(1+|\mathbf{D}|)$$

- Different upper and lower growth in principle - ψ has different polynomial upper and lower growth, for $\psi(\mathbf{D}) := \psi(|\mathbf{D}|)$

$$c_1|\mathbf{D}|^r - c_2 \le \psi(|\mathbf{D}|) \le c_3|\mathbf{D}|^q + c_4$$

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What is the goal?

- Goal = existence result for as general constitutive relationship as possible
- A priori = energy estimates (Ω bounded and sufficiently smooth, boundary conditions allowing to get the estimates)
  - Steady case
  $$\int_\Omega \psi(\mathbf{D}) + \psi^*(\mathbf{S})\, dx \le C$$
  - Unsteady case
  $$\sup_t \|v\|_2^2 + \int_0^T \int_\Omega \psi(\mathbf{D}) + \psi^*(\mathbf{S})\, dx\, dt \le C$$

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How to get the goal

- Energy equality "holds" $\Longrightarrow$ simpler proof, i.e., if

$$\boxed{\int (v \otimes v) : \mathbf{D}(v) \qquad \text{is meaningful}}$$

- More difficult case, i.e.,

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The key result

Theorem (Easier case; Gwiazda, Świerczewska-Gwiazda et al)

If energy equality "holds" and ψ^* satisfies Δ_2 conditions then there exists a weak solution for any relevant boundary conditions.

Theorem (Difficult case; Bulíček, Gwiazda, Málek and Świerczewska-Gwiazda)

Let $\psi(\mathbf{D}) := \psi(|\mathbf{D}|)$ and ψ and ψ^* satisfy Δ_2 condition. Assume that energy space is compactly embedded into L^2. Then there exists a weak solution for Navier's bc.

- The same result also holds for Dirichlet bc. by using the Wolf decomposition of the pressure.

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Byproducts-increase the citation report

Byproduct

Theory for the laplace equation with Neumann bc, i.e., for ψ and ψ^* satisfying Δ_2 condition, we have

$$\int_\Omega \psi(|\nabla^2 u|) \leq C \left(1 + \int_\Omega \psi(|f|)\right)$$

for any u solving homogeneous Neuman problem with right hand side f.

Byproduct

Improvement of the Minty method $\Longrightarrow$ no use of the Vitali theorem $\Longrightarrow$ no strict monotonicity required

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Improvement of the Lipschitz approximation method $\Longrightarrow$ no need of Δ_2 for ψ $\Longrightarrow$ nothing to our case due to the pressure $\Longrightarrow$ but may be use for general parabolic/elliptic problems

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Power-law like fluid - Explicit

Compact embedding is available if $r > \frac{6}{5}$

- $r = 2$ **Lerray** (1934)
- $r \geq \frac{11}{5}$ for unsteady, $r \geq \frac{9}{5}$ steady; **Ladyzhenskaya** 60's
- $r \geq \frac{9}{5}$ unsteady; **Málek. Nečas, Růžička** 90's
- $r \geq \frac{8}{5}$ unsteady; **Frehse, Málek, Steinahuer** (2000)
- $r > \frac{6}{5}$ steady; **Frehse, Málek, Steinahuer** (2002)
- $r > \frac{6}{5}$ unsteady; **Diening, Růžička, Wolf** (2009)

Power-law like fluid - implicit (discontinuous)

- $r \geq \frac{11}{5}$ - strict monotonicity - **Gwiazda, Málek, Świerczewska** (2007)
- $r > \frac{9}{5}$ - Herschel-Bulkley model - **Málek, Růžička, Shelukhin**(2005)
- $r > \frac{6}{5}$ steady - strict monotonicity - **Bulíček, Gwiazda, Málek, Świerczewska** (2009)
- $r > \frac{6}{5}$ unsteady; **Bulíček, Gwiazda, Málek, Świerczewska** (2010)

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- Fully implicit setting

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Lipschitz approximation

- sequence of solutions v^n; $v^n - v$ is not possible test function
- introduce a Lipschitz function $(v^n - v)_\lambda$ that is "closed" to to original
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Lipschitz approximation

Lemma

$\{u^n\}_{n=1}^{\infty}$ tends strongly to $\mathbf{0}$ in L^1 and $\{\mathbf{S}^n\}_{n=1}^{\infty}$ such that

$$\int_{\Omega} \psi^*(|\mathbf{S}^n|) + \psi(|\nabla u^n|)\, dx \le C^* \quad (C^* > 1).$$

Then for arbitrary $\lambda^* \in \mathbb{R}_+$ and $k \in \mathbb{N}$ there exists $\lambda^{\max} < \infty$ and there exists sequence of $\{\lambda_n^k\}_{n=1}^{\infty}$ and the sequence u_k^n (going to zero) and open sets $E_n^k := \{u_k^n \neq u^n\}$ such that $\lambda_n^k \in [\lambda^*, \lambda^{\max}]$ and for any sequence α_k^n

$$u_k^n \in W^{1,p}, \quad \|\mathbf{D}(u_k^n)\|_\infty \le C\lambda_n^k,$$

$$|\Omega \cap E_n^k| \le C\frac{C^*}{\psi(\lambda_n^k)},$$

$$\int_{\Omega\cap E_n^k} |\mathbf{S}^n \cdot \mathbf{D}(u_k^n)|\ dx \le CC^* \left(\frac{\alpha_n^k}{k} + \frac{\alpha_n^k \psi(\lambda_n^k/\alpha_n^k)}{\psi(\lambda_n^k)}\right)$$

Use of Lipschtiz approximation

- We have approximative problem $(v^n, \mathbf{S}^n)$ and weak limits $(v, \overline{\mathbf{S}})$, we need to show that $(\overline{\mathbf{S}}, \mathbf{D}(v)) \in \mathcal{A}$
- Test the approximative n- problem by Lipschitz approximation of $v^n - v$, i.e., $u^n_k := (v^n - v)_k$
- One gets (here $\mathbf{S}$ is such that $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$

$$\lim_{n\to\infty} \int_{u^n_k = u^n} (\mathbf{S}^n - \mathbf{S}) : \mathbf{D}(u^n_k) \le CC^* \left(\frac{\alpha^k_n}{k} + \frac{\alpha^k_n \psi(\lambda^k_n/\alpha^k_n)}{\psi(\lambda^k_n)} \right)$$

- Hölder inequality gives

$$\lim_{n\to\infty} \int_\Omega |(\mathbf{S}^n - \mathbf{S}) : \mathbf{D}(v^n - v)|^\varepsilon \le \int_{u^n = u^n_k} + \int_{u^n \neq u^n_k} \le \text{ small terms} \to 0$$

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- point-wise convergence of $(\mathbf{S}^n - \mathbf{S}) : \mathbf{D}(v^n - v)$ to 0; strict monotonicity finishes the proof
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$$(\mathbf{S}^n - \mathbf{S}) : \mathbf{D}(v^n - v) \text{ converges weakly in } L^1(\Omega \setminus A_k)$$

- point-wise & weak implies strong in $L^1(\Omega \setminus A_k)$
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$$(\overline{\mathbf{S}}, \mathbf{D}(v)) \in \mathcal{A} \quad \text{ for a.a. } x \in \Omega \setminus A_k$$

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Future?????

• Extension to whole N- function setting, i.e., ψ depends on whole $\mathbf{D}$ and not only on $|\mathbf{D}|$, very hard
• Extension to "real" x-dependent setting, i.e., the growth estimates depends crucially on x, i.e., for models

$$\mathbf{S} \sim (1 + |\mathbf{D}|)^{r(c(x))-2}\mathbf{D},$$

where c satisfy convection diffusion problem.