Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous dilute polymers

Endre Süli

Mathematical Institute, University of Oxford

joint work with

Miroslav Bulíček and **Josef Málek**

Charles University, Prague

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$: bounded open Lipschitz domain,
- T: length of the time interval of interest, and
- $Q := \Omega \times (0, T)$: the associated space-time domain.

Consider the following system of nonlinear PDEs:

$$\rho\,(\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})) - \operatorname{div}\mathbf{T} = \rho\,\boldsymbol{f} \qquad \text{in } Q, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } Q, \tag{2}$$

subject to the initial condition

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0(\cdot) \qquad \text{in } \Omega, \tag{3}$$

and the boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{4}$$

$$\lambda(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}} + (1 - \lambda)\gamma_* \boldsymbol{u}_{\boldsymbol{\tau}} = 0 \qquad \text{on } \partial\Omega \times (0, T). \tag{5}$$

Constitutive equation for the Cauchy stress tensor $\mathbf{T}$

We assume that the *Cauchy stress* $\mathbf{T}$ is decomposed as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}_v + \mathbf{S}_e, \tag{6}$$

where

- $p : Q \to \mathbb{R}$ is the pressure;
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  $\mathbf{S}_v$ and $\mathbf{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ are assumed to be related through a maximal monotone graph described by the implicit relation:

$$\mathbf{G}(\mathbf{S}_v, \mathbf{D}(\boldsymbol{u})) = \mathbf{0}, \tag{7}$$

  where $\mathbf{G} : \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is a continuous mapping.

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  Examples: [Rajagopal (2003, 2006), Rajagopal and Srinivasa (2008)]
  power-law fluids, stress power-law fluids, fluids with activation criteria (Bingham, Herschel–Bulkley), and shear-rate dependent fluids with discontinuous viscosities.

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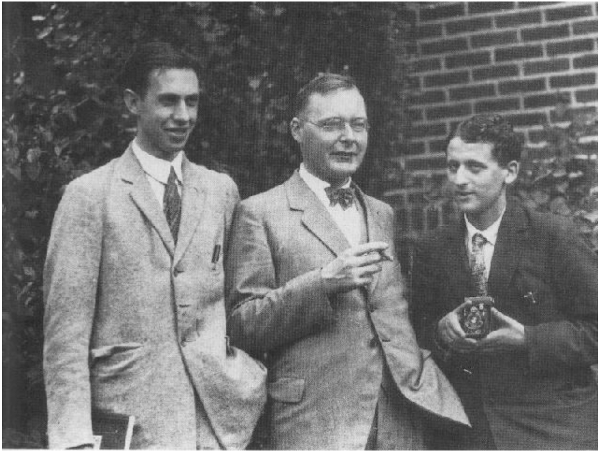
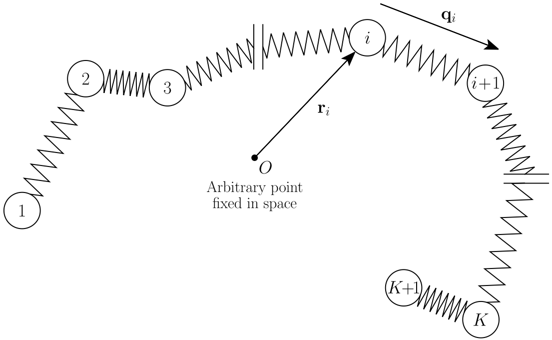
- $\mathbf{S}_e : Q \to \mathbb{R}^{d\times d}_{sym}$ is the elastic part of the stress.

Definition of $\mathbf{S}_e$: kinetic theory of polymers



George Uhlenbeck, Hans Kramers and Samuel Goudsmit
(Ann Arbor, Michigan – around 1928).

Definition of $\mathbf{S}_e$



Let $D_i \subset \mathbb{R}^d$, $i = 1, \ldots, K$, be bounded open balls centred at $\mathbf{0}$.

Consider the *Maxwellian* $M(\boldsymbol{q}) := M_1(\boldsymbol{q}_1) \cdots M_K(\boldsymbol{q}_K)$, with $\boldsymbol{q}_i \in D_i$, where

$$M_i(\boldsymbol{q}_i) := \frac{\mathrm{e}^{-U_i(\frac{1}{2}|\boldsymbol{q}_i|^2)}}{\int_{D_i} \mathrm{e}^{-U_i(\frac{1}{2}|\boldsymbol{p}_i|^2)} \mathrm{d}\boldsymbol{p}_i}, \qquad i = 1, \ldots, K.$$

S_e is defined by the *Kramers expression*:

$$\mathsf{S}_e(x,t) := k \sum_{i=1}^{K} \int_D M(\boldsymbol{q}) \, \nabla_{\boldsymbol{q}_i} \widehat{\psi}(x,\boldsymbol{q},t) \otimes \boldsymbol{q}_i \, \mathrm{d}\boldsymbol{q},$$

where $\boldsymbol{q} = (\boldsymbol{q}_1^{\mathrm{T}}, \ldots, \boldsymbol{q}_N^{\mathrm{T}})^{\mathrm{T}}$ and

$$\widehat{\psi} := \psi / M$$

is the renormalized probability density function.

Fokker–Planck equation

The probability density function satisfies the *Fokker–Planck equation*:

$$(M\widehat{\psi})_t + \operatorname{div}(M\widehat{\psi}\boldsymbol{u}) + \operatorname{div}_{\boldsymbol{q}}(M\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}) = \triangle(M\widehat{\psi}) + \operatorname{div}_{\boldsymbol{q}}\mathbb{A}(M\nabla_{\boldsymbol{q}}\widehat{\psi}) \quad (8)$$

in $O \times (0,T)$, with $O := \Omega \times D$, subject to the boundary conditions:

$$M\nabla\widehat{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\times D\times(0,T), \quad (9)$$

$$(M\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}_i - \mathbb{A}_i(M\nabla_{\boldsymbol{q}}\widehat{\psi}))\cdot\boldsymbol{n}_i = 0 \qquad \text{on } \Omega\times\partial\bar{D}_i\times(0,T), \quad (10)$$

for all $i = 1,\dots,K$, and the initial condition

$$\widehat{\psi}(x,\boldsymbol{q},0) = \widehat{\psi}_0(x,\boldsymbol{q}) \qquad \text{in } O. \quad (11)$$

$\mathbb{A}$: *Rouse matrix* (symmetric, positive definite).

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  Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains

- J.W. Barrett & E. Süli (M3AS, 22 (2012), 1–84):
  Existence and equilibration of global weak solutions to kinetic models for dilute polymers II: Hookean-type bead-spring chains

- J.W. Barrett & E. Süli (J. Diff. Eqs. submitted; arXiv:1112.4781, 2012; 1–70):
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M. Bulíček, J. Málek & E. Süli (Comm. PDE, 2012, submitted):
Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous flows of dilute polymers

Assumptions on the data

We identify the implicit relation (7) with a graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$, i.e.,

$$\mathbf{G}(\mathbf{S},\mathbf{D}) = \mathbf{0} \iff (\mathbf{S},\mathbf{D}) \in \mathcal{A}.$$

We assume that, for some $r \in (1,\infty)$, $\mathcal{A}$ is a *maximal monotone r-graph*:

(A1) *$\mathcal{A}$ includes the origin;* i.e., $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$;

(A2) *$\mathcal{A}$ is a monotone graph;* i.e.,

$$(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) \geq 0 \text{ for all } (\mathbf{D}_1,\mathbf{S}_1),(\mathbf{D}_2,\mathbf{S}_2) \in \mathcal{A};$$

(A3) *$\mathcal{A}$ is a maximal monotone graph;* i.e., for any $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$,

$$\text{if } (\tilde{\mathbf{S}} - \mathbf{S})\cdot(\tilde{\mathbf{D}} - \mathbf{D}) \geq 0 \text{ for all } (\tilde{\mathbf{D}},\tilde{\mathbf{S}}) \in \mathcal{A}, \text{ then } (\mathbf{D},\mathbf{S}) \in \mathcal{A};$$

(A4) *$\mathcal{A}$ is an r-graph;* i.e., there exist positive constants C_1, C_2 such that

$$\mathbf{S}\cdot\mathbf{D} \geq C_1(|\mathbf{D}|^r + |\mathbf{S}|^{r'}) - C_2 \text{ for all } (\mathbf{D},\mathbf{S}) \in \mathcal{A}.$$

For the Maxwellian M we assume that

$$M \in C(\overline{D}) \cap C^{0,1}_{\mathrm{loc}}(D) \cap W^{1,1}_0(D), \quad \text{and } M > 0 \text{ on } D. \tag{12}$$

For the initial velocity $\boldsymbol{u}_0$ we assume that

$$\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega). \tag{13}$$

For $\widehat{\psi}_0 := \psi_0/M$ we assume that

$$\widehat{\psi}_0 \geq 0 \ \text{ a.e. in } O, \quad \widehat{\psi}_0 \log \widehat{\psi}_0 \in L^1_M(O), \tag{14}$$

and in addition we require that

$$\int_D M(\boldsymbol{q})\, \widehat{\psi}_0(\cdot, \boldsymbol{q})\, \mathrm{d}\boldsymbol{q} \in L^\infty(\Omega). \tag{15}$$

Theorem

For $d \in \{2,3\}$ let $D_i \subset \mathbb{R}^d$, $i = 1, \ldots, K$, be bounded open balls centred at the origin in $\mathbb{R}^d$, let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and suppose $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}}(\Omega))$, $r \in (1,\infty)$. Assume that $\mathcal{A}$, given by **G**, *is a maximal monotone r-graph satisfying* **(A1) – (A4)**, *the Maxwellian $M : D \to \mathbb{R}$ satisfies* (12), *and $(\boldsymbol{u}_0, \widehat{\psi}_0)$ satisfy* (13)–(15).

Then, there exist $(\boldsymbol{u}, \mathbf{S}_v, \mathbf{S}_e, \widehat{\psi})$ such that

$$\boldsymbol{u} \in L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega)^d) \cap L^r(0,T;W^{1,r}_0(\Omega)^d) \cap W^{1,r^*}(0,T;W^{-1,r^*}_{0,\mathrm{div}}(\Omega)),$$

$$\mathbf{S}_v \in L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), \qquad \mathbf{S}_e \in L^2(0,T;L^2(\Omega)^{d\times d}),$$

$$\widehat{\psi} \in L^\infty(Q;L^1_M(D)) \cap L^2(0,T;W^{1,1}_M(O)), \qquad \widehat{\psi} \geq 0 \text{ a.e. in } O \times (0,T),$$

$$M\widehat{\psi} \in W^{1,1}(0,T;W^{-1,1}(O)), \qquad \widehat{\psi}\log\widehat{\psi} \in L^\infty(0,T;L^1_M(O)),$$

where

$$r^* := \min\left\{r', 2, \left(1+\tfrac{2}{d}\right)r\right\} \qquad \text{and} \qquad r' := \frac{r}{r-1}.$$

Theorem (Continued...)

Moreover, (1) *is satisfied in the following sense:*

$$\int_0^T \langle \boldsymbol{u}_t, \boldsymbol{w} \rangle \,\mathrm{d}t + \int_0^T \big(-(\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{w}) + (\mathsf{S}_v, \nabla \boldsymbol{w}) \big) \mathrm{d}t$$
$$= \int_0^T \big(-(\mathsf{S}_e, \nabla \boldsymbol{w}) + \langle \boldsymbol{f}, \boldsymbol{w} \rangle \big) \mathrm{d}t \qquad \text{for all } \boldsymbol{w} \in L^\infty(0,T; W^{1,\infty}_{0,\mathrm{div}}(\Omega)),$$

where

$$(\mathsf{S}_v(x,t), \mathsf{D}(\boldsymbol{u}(x,t))) \in \mathcal{A} \qquad \text{for a.e. } (x,t) \in Q,$$

and S_e *is given by the Kramers expression*

$$\mathsf{S}_e(x,t) = k \sum_{i=1}^{K} \int_D M \nabla_{\boldsymbol{q}_i} \widehat{\psi}(x, \boldsymbol{q}, t) \otimes \boldsymbol{q}_i \,\mathrm{d}\boldsymbol{q} \quad \text{for a.e. } (x,t) \in Q.$$

Theorem (Continued...)

In addition, the Fokker–Planck eqn (8) *is satisfied in the following sense:*

$$\int_0^T \left[\langle (M\widehat{\psi})_t, \varphi\rangle - (M\,\boldsymbol{u}\widehat{\psi}, \nabla\varphi)_{\mathcal{O}} - (M\,\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}\right] \mathrm{d}t$$
$$+ \int_0^T \left[(M\,\nabla\widehat{\psi}, \nabla\varphi)_{\mathcal{O}} + (M\,\mathbb{A}\nabla_{\boldsymbol{q}}\widehat{\psi}, \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}\right] \mathrm{d}t = 0$$
$$\textit{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})),$$

and the initial data are attained strongly in $L^2(\Omega)^d \times L^1_M(\mathcal{O})$*, i.e.,*

$$\lim_{t\to 0_+} \|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0(\cdot)\|_2^2 + \|\widehat{\psi}(\cdot,t) - \widehat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})} = 0.$$

Theorem (Continued...)

Further, for $t \in (0,T)$ the following energy inequality holds in a weak sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_O kM\widehat{\psi}\log\widehat{\psi}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\frac{1}{2}\|\boldsymbol{u}\|_2^2\right)+(\mathsf{S}_v,\mathsf{D}(\boldsymbol{u}))+4k\left(M\nabla\sqrt{\widehat{\psi}},\nabla\sqrt{\widehat{\psi}}\right)_O$$

$$+4k\left(M\,\mathbb{A}\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}},\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}}\right)_O\leq\langle\boldsymbol{f},\boldsymbol{u}\rangle.$$

Proof

STEP 1. Truncate $\widehat{\psi}$ in the Kramers expression and in the drag term in the FP equation by replacing $\widehat{\psi}$ with $T_\ell(\widehat{\psi})$, preserving the energy inequality.

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STEP 5. We require strongly convergent sequences for passage to limit in ℓ in the various nonlinear terms. This is the most difficult step to realize.

weak convergence $\longrightarrow$ strong convergence

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  strong convergence immediate by Aubin–Lions–Simon compactness theorem.

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  - Murat–Tartar Div–Curl lemma;

weak convergence $\longrightarrow$ strong convergence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_O kM\widehat{\psi}^\ell \log\widehat{\psi}^\ell \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{u}^\ell\|_2^2\right) + (\mathbf{S}_\nu^\ell, \mathbf{D}(\boldsymbol{u}^\ell)) + 4k\left(M\nabla\sqrt{\widehat{\psi}^\ell}, \nabla\sqrt{\widehat{\psi}^\ell}\,\right)_O$$
$$+\,4k\left(M\,\mathbb{A}\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}^\ell}, \nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}^\ell}\,\right)_O \leq \langle \boldsymbol{f}, \boldsymbol{u}^\ell\rangle.$$

- Velocity:

  strong convergence immediate by Aubin–Lions–Simon compactness theorem.

- Probability density function: (much more difficult)

  Idea 1: Dubinskiĭ's extension of the Aubin–Lions–Simon theorem

  Idea 2:
  - Vitali's convergence theorem (a.e. convergence + L_1 equi-integrability);
  - Weak lower semicontinuity of convex functions (Feireisl & Novotný);
  - Murat–Tartar Div–Curl lemma;
  - Uniform interior estimates on $\Omega \times D \times (0,T)$, obtained by function space interpolation from the energy inequality.

STEP 6. The sequence of truncated Kramers expressions $\mathbf{S}_e^\ell$ converges to $\mathbf{S}_e$ strongly in $L^q(0,T;L^q(\Omega)^{d\times d})$ for all $q\in[1,2)$.

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STEP 7. The initial data are attained strongly in $L^2(\Omega)^d \times L^1_M(O)$, i.e.,

$$\lim_{t\to 0_+} \|\boldsymbol{u}(\cdot,t)-\boldsymbol{u}_0(\cdot)\|_2^2 + \|\widehat{\psi}(\cdot,t)-\widehat{\psi}_0(\cdot)\|_{L^1_M(O)} = 0.$$

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STEP 8. Identification of $\mathbf{S}_v$: noting the strong convergence of $\mathbf{S}_e^\ell$ and $\boldsymbol{u}^\ell$,

- we use the method of parabolic Lipschitz-truncation
  (Diening, Ružička & Wolf (2010)), and
- Chacon's biting lemma

to finally deduce that

$$(\mathbf{S}_v, \mathbf{D}(\boldsymbol{u})) \in \mathcal{A} \qquad \text{for a.e. } (x,t) \in Q.$$

Bulíček, Gwiazda, Málek & Świerczewska-Gwiazda:
SIAM J. Math. Anal. (Accepted). Preprint of NCMM, no. 2011-008, 2011.

Summary

We have established long-time large-data existence of weak solutions to a general class of kinetic models of homogeneous incompressible dilute polymers, the main new feature of the model being the presence of a general implicit constitutive equation relating the viscous part $\mathbf{S}_v$ of the Cauchy stress and the symmetric part $\mathbf{D}$ of the velocity gradient.

The elastic properties of the flow, characterizing the response of polymer macromolecules in the viscous solvent, have been modelled by the elastic part $\mathbf{S}_e$ of the Cauchy stress tensor, which is defined by the Kramers expression involving the probability density function, associated with the random motion of the polymer molecules in the solvent.

The probability density function satisfies a Fokker–Planck equation, which is nonlinearly coupled to the momentum equation.

Possible extensions

A possible extension of the analysis presented here would be to admit a nonhomoheneous solvent, with variable density.

For a coupled Navier–Stokes–Fokker–Planck system with variable density and density-dependent dynamic viscosity and drag coefficients the existence of global weak solutions was shown by

- Barrett & Süli (2012).

The main theoretical hurdle in extending the results of Barrett & Süli (2012) to nonhomogeneous fluid flow models where instead of a linear relationship between $\mathbf{S}_\nu$ and $\mathbf{D}$ these quantities are related through an implicit relationship is that the parabolic Lipschitz-truncation method of

- Diening, Ružička & Wolf (2010), and
- Bulíček, Gwiazda, Málek & Świerczewska-Gwiazda (2011)

is not (yet) available for such models.