Navier-Stokes-Cahn-Hilliard system with chemotaxis

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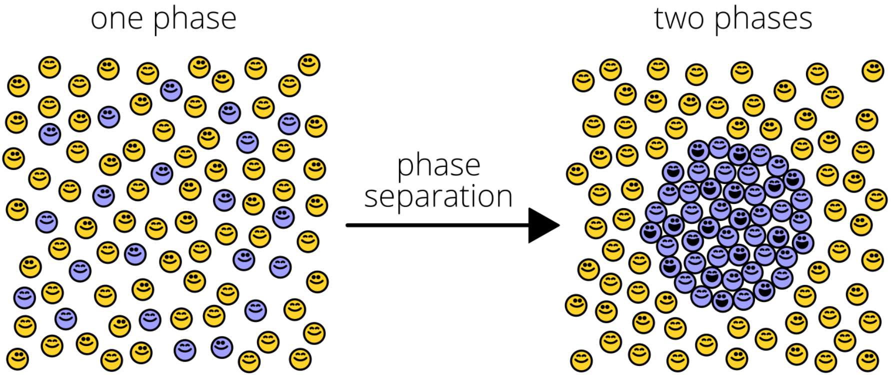
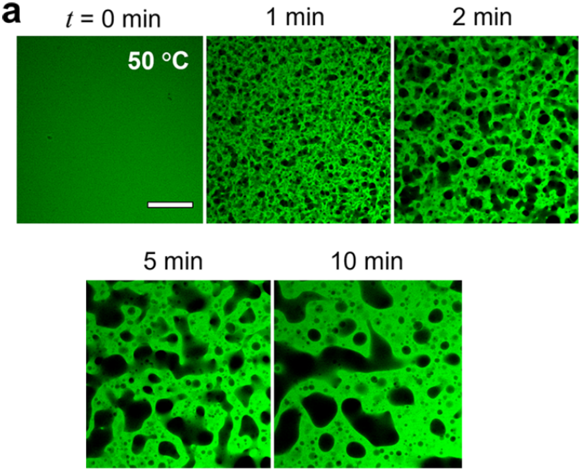
Mixtures: Modeling, analysis and computing
Prague, 5th–7th February 2025

Phase separation



Credits to Drummond Lab

Phase separation in polymer mixtures



Viscoelastic phase separation in PEP hydrogels.

Rapp & Silverman, Macromolecules 2022

Model H: Navier-Stokes-Cahn-Hilliard system

AIM: Phase separation for a mixture of two incompressible viscous fluids

State variables: $\boldsymbol{u}$ = averaged velocity, P =pressure

ϕ = difference of concentrations $\in [-1, 1]$

$$
\begin{aligned}
&\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}\left(\nu(\phi) D\boldsymbol{u}\right) + \nabla P = -\varepsilon \operatorname{div}\left(\nabla\phi \otimes \nabla\phi\right) \\
&\operatorname{div} \boldsymbol{u} = 0 \\
&\partial_t \phi + \boldsymbol{u} \cdot \nabla\phi = \operatorname{div}\left(m(\phi)\nabla\mu\right) \\
&\mu = -\varepsilon\Delta\phi + \frac{1}{\varepsilon}\Psi'(\phi)
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Hohenberg & Halpering, Rev. Mod. Phys. 1977

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Hohenberg & Halpering, Rev. Mod. Phys. 1977

Boundary and initial conditions: Ω bounded smooth set in $\mathbb{R}^d$, $d = 2, 3$

$$
\boldsymbol{u} = \mathbf{0},\ \partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \phi(0) = \phi_0 \quad \text{in } \Omega
$$

Abels-Garcke-Grün (AGG) model

$$\partial_t(\rho(\phi)\boldsymbol{u}) + \operatorname{div}\big(\boldsymbol{u} \otimes (\rho(\phi)\boldsymbol{u} + \mathbf{J})\big) - \operatorname{div}(\nu(\phi)D\boldsymbol{u}) + \nabla P = -\varepsilon \operatorname{div}(\nabla\phi \otimes \nabla\phi)$$

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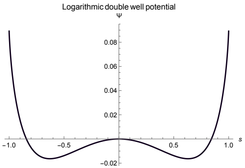
$$\mathbf{J} = -\frac{\rho_1 - \rho_2}{2} m(\phi)\nabla\mu, \qquad \rho(\phi) = \rho_1 \frac{1+\phi}{2} + \rho_2 \frac{1-\phi}{2}, \qquad \nu(\phi) = \nu_1 \frac{1+\phi}{2} + \nu_2 \frac{1-\phi}{2}$$

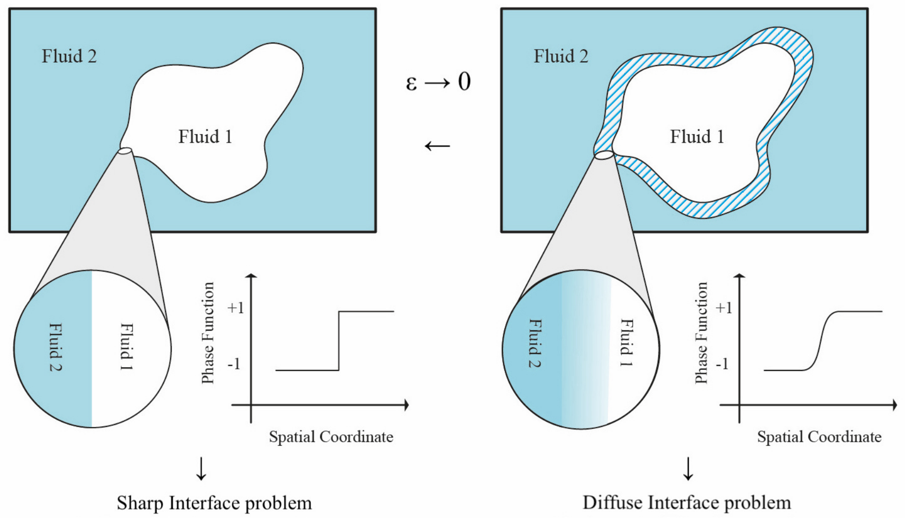
$$\Psi(\phi) = \frac{\theta}{2}\Big[(1+\phi)\ln(1+\phi) + (1-\phi)\ln(1-\phi)\Big] - \frac{\theta_0}{2}\phi^2$$

$$m(\phi) = \varepsilon \text{ or } 1-\phi^2, \quad D = \frac{1}{2}(\nabla + \nabla^T), \quad \varepsilon > 0$$



Abels, Garcke & Grün , M3AS 2012

Mathematical description of phase separation



Two-phase Navier-Stokes equations as $\varepsilon \to 0$

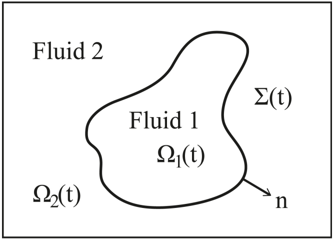
$\Omega = \Omega_1(t) \cup \Omega_2(t) \cup \Sigma(t)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $t \geq 0$

$$\begin{aligned} \rho_1 \partial_t \boldsymbol{u}_1 + \rho_1 \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 - \nu_1 \mathrm{div}\,(D\boldsymbol{u}_1) + \nabla p_1 = 0, \mathrm{div}\, \boldsymbol{u}_1 = 0, &\quad \text{in } \Omega_1(t) \\ \rho_2 \partial_t \boldsymbol{u}_2 + \rho_2 \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u}_2 - \nu_2 \mathrm{div}\,(D\boldsymbol{u}_2) + \nabla p_2 = 0, \mathrm{div}\, \boldsymbol{u}_2 = 0, &\quad \text{in } \Omega_2(t) \end{aligned}$$

subject to

$$\begin{cases} \boldsymbol{u}_1 = \boldsymbol{u}_2, & \text{on } \Sigma(t) \\ (T_1 - T_2)\boldsymbol{n} = \sigma H \boldsymbol{n}, & \text{on } \Sigma(t) \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0(\cdot) & \text{in } \Omega \end{cases}$$

where $T_i = \nu_i D\boldsymbol{u}_i - p_i I$, H =mean curvature



Properties of the AGG system

- **Energy equation**

$$E(\boldsymbol{u},\phi) = \underbrace{\int_\Omega \frac{1}{2}\rho(\phi)|\boldsymbol{u}|^2 \,\mathrm{d}x}_{\text{kinetic energy}} + \underbrace{\int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x}_{\text{free energy}}$$

$$\Downarrow$$

$$E(\boldsymbol{u}(t),\phi(t)) + \int_0^t \int_\Omega \nu(\phi)|D\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega m(\phi)|\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}\tau = E(\boldsymbol{u}_0,\phi_0), \quad \forall\, t \geq 0$$

- **Conservation of mass**

$$\overline{\phi}(t) = \frac{1}{|\Omega|}\int_\Omega \phi(t)\,\mathrm{d}x = \overline{\phi_0}, \quad \forall\, t \geq 0$$

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- **Equivalent formulation**

$$\partial_t\rho(\phi) + \mathrm{div}(\rho(\phi)\boldsymbol{u} + \mathbf{J}) = 0$$

$$\Downarrow$$

$$\rho(\phi)\partial_t\boldsymbol{u} + \rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \rho'(\phi)(\nabla\mu\cdot\nabla)\boldsymbol{u} - \mathrm{div}(\nu(\phi)D\boldsymbol{u}) + \nabla P = -\mathrm{div}(\nabla\phi\otimes\nabla\phi)$$

Global weak solutions to the AGG model

Theorem 1 (Global existence of weak solutions)

Let m be non-degenerate. Assume that $\boldsymbol{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$, $\phi_0 \in H^1(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)} \leq 1$ and $|\overline{\phi_0}| < 1$. Then, there exists a ***global weak solution*** $(\boldsymbol{u}, \phi)$ *on* $\Omega \times [0, \infty)$ *such that*

$$\boldsymbol{u} \in C_{\mathrm{w}}([0,\infty); \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,\infty; \mathbf{H}^1_{0,\sigma}(\Omega)),$$

$$\phi \in C_{\mathrm{w}}([0,\infty); H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); H^2(\Omega)), \quad \Psi'(\phi) \in L^2_{\mathrm{uloc}}(0,\infty; L^2(\Omega)),$$

$$\phi \in L^\infty(\Omega \times (0,\infty)) : \quad |\phi(x,t)| < 1 \quad \textit{a.e. in } \Omega \times (0,\infty),$$

$$\mu \in L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega)), \quad \nabla\mu \in L^2(0,\infty; L^2(\Omega)),$$

which satisfies the AGG system in weak sense. In addition, the energy inequality

$$E(\boldsymbol{u}(t), \phi(t)) + \int_s^t \|\sqrt{\nu(\phi)} D\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\sqrt{m(\phi)}\nabla\mu\|^2_{L^2(\Omega)} \, \mathrm{d}\tau \leq E(\boldsymbol{u}(s), \phi(s))$$

holds for all $t \in [s, \infty)$ and almost all $s \in [0,\infty)$ (including $s = 0$).

Abels, Depner & Garcke, JMFM 2013 - AIHP ANL 2013

Main result: regularity and stabilization

Theorem 2 (Abels, Garcke & G., Math. Ann. 2024)

Let Ω be a bounded domain in $\mathbb{R}^3$ and $m \equiv 1$. Consider a global weak solution $(\boldsymbol{u}, \phi)$ on $\Omega \times [0, \infty)$. Then, we have:

(i) ***Global regularity of the concentration****: for any $\tau > 0$, we have*

$$\phi \in L^\infty(\tau, \infty; W^{2,6}(\Omega)), \quad \partial_t \phi \in L^2(\tau, \infty; H^1(\Omega)),$$
$$\mu \in L^\infty(\tau, \infty; H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([\tau, \infty); H^3(\Omega)), \quad \Psi'(\phi) \in L^\infty(\tau, \infty; L^6(\Omega)).$$

In addition, there exists $C > 0$ such that

$$\|\nabla \mu\|^2_{L^\infty(\tau,\infty;L^2)} + \int_\tau^\infty \|\nabla \partial_t \phi(s)\|^2_{L^2} \, \mathrm{d}s + \int_\tau^\infty \|\nabla \mu(s)\|^2_{H^2} \, \mathrm{d}s$$
$$\leq C \left(\left\|\nabla(-\Delta \phi(\tau) + \Psi'(\phi(\tau)))\right\|^2_{L^2} + \int_\tau^\infty \|\nabla \boldsymbol{u}(s)\|^2_{L^2} + \|\nabla \mu(s)\|^2_{L^2} \, \mathrm{d}s \right)$$
$$\times \exp\left(C \int_\tau^\infty \|\nabla \boldsymbol{u}(s)\|^2_{L^2} \, \mathrm{d}s \right).$$

Main result 2: regularity and stabilization

(ii) ***Separation property***: *there exist* $T_{SP} > 0$ *and* $\delta > 0$ *such that*

$$|\phi(x,t)| \leq 1-\delta, \quad \forall\,(x,t) \in \overline{\Omega} \times [T_{SP}, \infty).$$

(iii) ***Large time regularity of the velocity***: *there exists* $T_R > 0$ *such that*

$$\boldsymbol{u} \in L^{\infty}(T_R,\infty;\mathbf{H}^1_{0,\sigma}(\Omega)) \cap L^2(T_R,\infty;\mathbf{H}^2(\Omega)) \cap H^1(T_R,\infty;\mathbf{L}^2_{\sigma}(\Omega)).$$

(iv) ***Convergence to equilibrium***: $(\boldsymbol{u}(t),\phi(t)) \to (\mathbf{0},\phi_\infty)$ *in* $\mathbf{L}^2(\Omega) \times W^{2-\epsilon,6}(\Omega)$ *as* $t \to \infty$, *for any* $\epsilon > 0$, *where* $\phi_\infty \in W^{2,p}(\Omega)$, *such that* $\overline{\phi_\infty} = \overline{\phi_0}$, *is a solution to the stationary Cahn-Hilliard equation*

$$\begin{aligned} -\Delta\phi_\infty + \Psi'(\phi_\infty) &= \overline{\Psi'(\phi_\infty)} && \text{in } \Omega, \\ \partial_{\boldsymbol{n}}\phi_\infty &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Non-degenerate mobility: $m(\phi) > 0$

Conti, Galimberti, Gatti & G., CVPDE 2025

NSCH model with chemotaxis

σ = concentration of (massless) chemical substance (e.g. nutrient of tumor cells)

$$\partial_t\left(\rho(\phi)\boldsymbol{u}\right) + \operatorname{div}\left(\boldsymbol{u}\otimes\left(\rho(\phi)\boldsymbol{u}+\mathbf{J}\right)\right) - \operatorname{div}\left(\nu(\phi)D\boldsymbol{u}\right) + \nabla P = \underbrace{-\operatorname{div}\left(\nabla\phi\otimes\nabla\phi\right)}_{=\nabla(\dots)+\mu\nabla\phi+w\nabla\sigma}$$

$$\operatorname{div}\boldsymbol{u} = 0$$

$$\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi = \Delta\mu$$

$$\mu = -\Delta\phi + \Psi'(\phi) + \chi\sigma$$

$$\partial_t\sigma + \boldsymbol{u}\cdot\nabla\sigma - \operatorname{div}\left(\sigma\nabla w\right) = 0$$

$$w = \ln\sigma + \chi\phi$$

Abels, Garcke & Grün , M3AS 2012

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Abels, Garcke & Grün , M3AS 2012

Logistic source: $\beta(\varphi)\sigma - \kappa\sigma^2$

Rocca, Schimperna & Signori, JDE 2023; Agosti & Signori, JDE 2024; G., He & Wu, arXiv 2024

Energy and conserved quantities

- **Energy equation**

$$E(\boldsymbol{u}, \phi, \sigma) = \int_\Omega \frac{1}{2}\rho(\phi)|\boldsymbol{u}|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi) + \sigma\,(\ln\sigma - 1) + \chi\phi\sigma \,\mathrm{d}x$$

$$\Downarrow$$

$$E(\boldsymbol{u}(t), \phi(t), \sigma(t)) + \int_0^t \int_\Omega \nu(\phi)|D\boldsymbol{u}|^2 + |\nabla\mu|^2 + \sigma|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}s = E(\boldsymbol{u}_0, \phi_0, \sigma_0), \quad \forall\, t \geq 0$$

- **Conservation of mass**

$$\overline{\phi}(t) = \frac{1}{|\Omega|}\int_\Omega \phi(t)\,\mathrm{d}x = \overline{\phi_0}, \quad \forall\, t \geq 0$$

$$\|\sigma(t)\|_{L^1(\Omega)} = \int_\Omega \sigma(x,t)\,\mathrm{d}x = \int_\Omega \sigma_0(x)\,\mathrm{d}x = \|\sigma_0\|_{L^1(\Omega)}, \quad \forall\, t \geq 0.$$

Result: Global existence of regular solutions

Theorem 3 (G., He & Wu, 2025)

Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Assume that

$$\boldsymbol{u}_0 \in \mathbf{H}^1_{0,\sigma}(\Omega), \quad \phi_0 \in H^2(\Omega) \text{ with } \|\phi_0\|_{L^\infty(\Omega)} \le 1, |\overline{\phi_0}| < 1,$$
$$\text{and } -\Delta\phi_0 + \Psi'(\phi_0) \in H^1(\Omega), \sigma_0 \in H^1(\Omega) \text{ such that } \sigma_0 \ge 0 \text{ a.e. in } \Omega.$$

Then, there exists a global strong solution $(\boldsymbol{u}, \phi, \sigma)$ defined on $\Omega \times [0,\infty)$ such that, for any $T > 0$,

$$\boldsymbol{u} \in L^\infty([0,T]; \mathbf{H}^1_{0,\sigma}(\Omega)) \cap L^2(0,T; \mathbf{H}^2(\Omega)),$$
$$\phi \in L^\infty(0,T; W^{2,p}(\Omega)) \text{ with } |\phi(x,t)| < 1 \text{ a.e. in } \Omega \times (0,\infty),$$
$$\mu \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^3(\Omega)), \Psi'(\phi) \in L^\infty(0,T; L^p(\Omega)),$$
$$\sigma \in L^\infty(0,T; L^p(\Omega)) \cap L^2(0,T; H^1(\Omega)) \text{ with } \sigma(x,t) \ge 0 \text{ a.e. in } \Omega \times (0,\infty),$$
$$\partial_t\sigma \in L^2(0,T; H^1(\Omega)'), \quad \sqrt{\sigma}\nabla w \in L^2(0,\infty; L^2(\Omega)),$$

for any $p \in [2,\infty)$.

Thank you!

H. Abels, H. Garcke, A. Giorgini,
Global regularity and asymptotic stabilization for the incompressible Navier-Stokes-Cahn-Hilliard model with unmatched densities,
Mathematische Annalen **389** (2024), 267–1321.

M. Conti, P. Galimberti, S. Gatti, A. Giorgini,
New results for the Cahn-Hilliard equation with non-degenerate mobility: well-posedness and longtime behavior,
to appear in Calc. Var. Partial Differ. Equ., (2025).

A. Giorgini, J. He & H. Wu,
Global Weak Solutions to a Navier–Stokes–Cahn–Hilliard System with Chemotaxis and Mass Transport: Cross Diffusion versus Logistic Degradation,
arXiv:2412.05751, (2024).