Analysis and Numerics of Viscoelastic Phase Separation

M. Lukáčová -Medvid'ová

Johannes Gutenberg University of Mainz

collaboration with: **A. Brunk, B. Dünweg**

- Typical model: Cahn-Hilliard-Navier-Stokes system (Model H)

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\begin{aligned}
\frac{\partial \phi}{\partial t} + \quad \mathbf{u} \cdot \nabla \phi &= \operatorname{div}\big(\phi^2(1-\phi^2)\nabla \mu\big) \\
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\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \operatorname{div}\big(\eta(\phi)\mathrm{D}\mathbf{u}\big) - \nabla p + \nabla \phi \mu \\
\operatorname{div}(\mathbf{u}) &= 0
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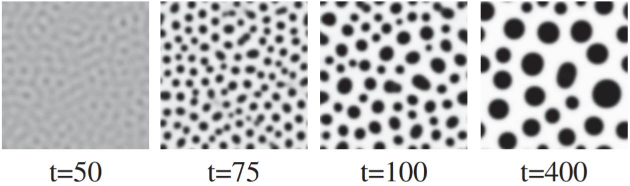

Figure: Simulation: Standard phase separation model

Phase separation of a polymeric fluid

Replacing one component by a polymer introduces new effects (transient gel)

[1] *Tanaka: Viscoelastic phase separation, J. Condens. Matter Phys., 2000*

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- Dynamic asymmetry of the components (slow and fast components)
- Relaxation effects for polymer and solvent on different timescales
  viscoelastic relaxation in pattern formation $\Rightarrow$
- Transient network structure of slow phase
  $\Rightarrow$ Flow through a porous medium (polymer network)
- Tanaka[1] proposed a concept of viscoelastic phase separation

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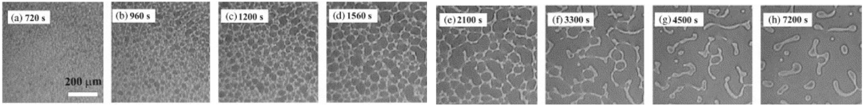
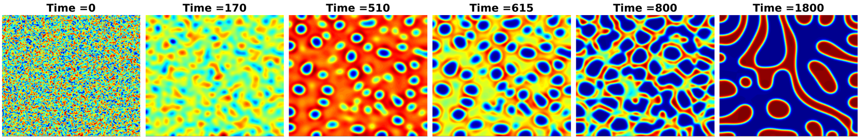
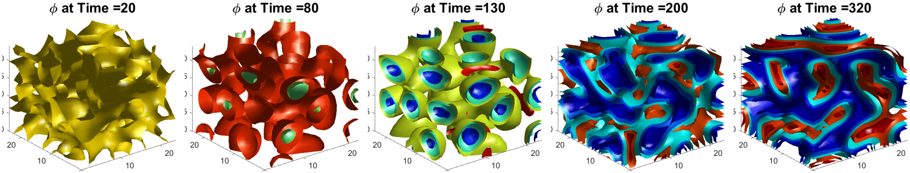
Experiments(PS/PVME) and simulations







Top: Tanaka's real experiments, Mid: 2D simulation, Bottom: 3D simulations via variational methods.

Literature overview

- Elliott, Garcke ('96) . . . CH with degenerate mobilities
- Ables ('09) . . . CH-NS
- Abels, Depner, Garcke ('13) . . . CH-NS with degenerate mobilities
- Abels, Garcke, Grün ('12) . . . two-phase model, different densities
- Cancés et al. ('19) . . . CH as constrained Wasserstein gradient flow
- Grün, Metzger ('16) . . . micro-macro models two-phase, dilute polymers

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- Barrett, Boyaval ('09) ... existence, regularized Oldroyd-B
- Barrett, Süli ('12-'18) ... existence diffusive Oldroyd-B
- Lei, Masmoudi. Zhou ('05) ... blow-up criterion
- Chupin ('18) ... strong sols.
- Constantin, Kriegl ('12) ... global regularity in 2D
- Geissert et al. ('12) ... strong sols.
- Lu, Zhang ('18) ... relative energy
- Zhang, Fang ('12) ... global strong sol. in critical L^p spaces
- Bathory, Bulíček, Málek ('24) ... existence, NSF & Johnson-Segalman stress-diffusive viscoelastic model
- Bulíček, Málek, Průša, Süli ('21) ... incomp. heat-conducting stress-diffusive rate-type model
- Bulíček, Los, Málek ('25) ... 3D Giesekus models

Macro Model

- ϕ polymer volume fraction ($(1-\phi)$ solvent volume fraction)
- q bulk stress related to polymer effects
- $\boldsymbol{\sigma}$ viscoelastic stress
- $\mathbf{u}$ volume-averaged velocity of polymers and solvents

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Variational principle: minimizing the total energy

$$E = F(\phi) + \int \frac{1}{2} q^2 + \int \frac{1}{2} \mathsf{Tr}(\sigma) + \int \frac{1}{2} |u|^2, \qquad \frac{d}{dt} E \leq 0$$

- $F(\phi)$ ... free energy

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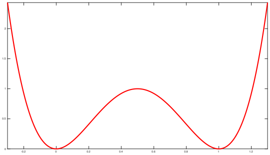
- $F(\phi)$ ... free energy

$$F(\phi) = \int f(\phi) + \frac{\gamma}{2} |\nabla \phi|^2 dx$$



Viscoelastic phase separation I [Zhou, Zhang and E ('06)]

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- $\boldsymbol{\sigma}$ viscoelastic stress
- $\mathbf{u}$ volume-averaged velocity of polymers and solvents

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\begin{aligned}
\partial_t \phi + \quad \mathbf{u}\cdot\nabla\phi &= \operatorname{div}\Big(\overbrace{\phi^2(1-\phi)^2}^{m(\phi)}\nabla\mu\Big) - \operatorname{div}\Big(\overbrace{\phi(1-\phi)}^{n(\phi)}\nabla\big(A(\phi)q\big)\Big)\\
\mu &= -c_0\Delta\phi + f'(\phi)\\
\partial_t q + \quad \mathbf{u}\cdot\nabla q &= -\frac{1}{\tau_b(\phi)}q + A(\phi)\Delta\big(A(\phi)q\big) - A(\phi)\operatorname{div}\Big(\overbrace{\phi(1-\phi)}^{n(\phi)}\nabla\mu\Big)\\
\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\big(\eta(\phi)\mathrm{D}\mathbf{u}\big) - \nabla p + \operatorname{div}\boldsymbol{\sigma} - \operatorname{div}\big(\gamma\nabla\phi\otimes\nabla\phi\big)\\
\operatorname{div}(\mathbf{u}) &= 0\\
\partial_t\boldsymbol{\sigma} + (\mathbf{u}\cdot\nabla)\boldsymbol{\sigma} &= (\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^\top - \frac{1}{\tau_s(\phi)}\boldsymbol{\sigma} + B(\phi)\mathrm{D}\mathbf{u}
\end{aligned}
$$

$$
E(\phi,q,\mathbf{u},\boldsymbol{\sigma}) = \int \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}|q|^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}).
$$

Viscoel. phase separation II [Brunk, ML (Nonlinerity '22)]

$$
\begin{aligned}
\partial_t \phi + \quad \mathbf{u} \cdot \nabla \phi &= \operatorname{div}\big(m(\phi)\nabla \mu\big) - \operatorname{div}\Big(n(\phi)\nabla\big(A(\phi)q\big)\Big)\\
\mu &= -\gamma \Delta \phi + f'(\phi)\\
\partial_t q + \quad \mathbf{u} \cdot \nabla q &= -\frac{1}{\tau(\phi)} q + A(\phi)\Delta\big(A(\phi)q\big) - A(\phi)\operatorname{div}\big(n(\phi)\nabla\mu\big) + \varepsilon_1 \Delta q\\
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\big(\eta(\phi)\mathrm{D}\mathbf{u}\big) - \nabla p + \operatorname{div}\operatorname{tr}(\mathbf{C})\mathbf{C} - \operatorname{div}\big(\gamma \nabla\phi \otimes \nabla\phi\big)\\
\operatorname{div}(\mathbf{u}) &= 0\\
\partial_t \mathbf{C} + (\mathbf{u}\cdot\nabla)\mathbf{C} &= (\nabla \mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top - h(\phi)\operatorname{tr}(\mathbf{C})\left[\operatorname{tr}(\mathbf{C})\mathbf{C} - \mathbf{I}\right] + \varepsilon_2 \Delta \mathbf{C}
\end{aligned}
$$

- viscoelastic stress tensor $\quad \mathbf{T} = \operatorname{tr}(\mathbf{C})\,\mathbf{C} - \mathbf{I}$

$$
E(\phi, q, \mathbf{u}, \mathbf{C}) = \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}|q|^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}\operatorname{tr}\left(\operatorname{tr}(\mathbf{T}) - 2\ln \mathbf{C}\right)
$$

$$
E(t) + \int_0^t D(s)\, ds \le E(0)
$$

- additional energy functional (Lyapunov functional)

$$E = \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}|q|^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}|\mathbf{C}|^2 \ \mathrm{dx}$$

with energy dissipation

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$$E(t) + \int_0^t \int_\Omega \left(\sqrt{m(\phi)}\nabla\mu - \nabla(A(\phi)q)\right)^2 + \varepsilon_1|\nabla q|^2 + \eta(\phi)|\mathrm{D}(\mathbf{u})|^2$$

$$+ \int_0^t \int_\Omega \frac{q^2}{\tau(\phi)} + \frac{\varepsilon_2}{2}|\nabla\mathbf{C}|^2 + \frac{1}{2}h(\phi)\mathrm{tr}(\mathbf{C})^2|\mathbf{C}|^2 - \frac{1}{2}h(\phi)\mathrm{tr}(\mathbf{C})^2 \; = E(0)$$

Well-posedness

▶ Well-posedness

1. Existence of solutions
2. Uniqueness of solution
3. Solution depends continuously on data

Well-posedness

▶ Well-posedness

1. Existence of solutions $\longrightarrow$ *weak solutions*
2. Uniqueness of solution
3. Solution depends continuously on data

▶ Existence [Brunk, M.L. (Nonlinearity 2022)]

There exists at least one global weak dissipative solution, in *two space dimensions*, which satisfies the integrated energy inequality.

Existence/ regular & singular case

Regular case

- All parametric functions are continuous and positively bounded
- $A(s)$ is allowed to be zero somewhere
- mobility functions $m(s), n(s)$

$$0 < m_1 \leq m(s) = n(s) \leq 1,$$

- mixing potential $f(s)$ is polynomial (Ginzburg-Landau) potential, e.g. $f(s) = (s-1)^2 s^2$

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Singular case

- mobility functions $m(s), n(s) = 0$ if and only if $x \in \{0, 1\}$

$$m(s) = n(s)^2$$

- mixing potential $f(s)$ is logarithmic (Flory-Huggins) potential

Weak solutions

▶ Brunk, ML (Nonlinearity '22)

For given initial conditions

$$(\phi_0, q_0, \mathbf{u}_0, \mathbf{C}_0) \in \left[H^1(\Omega) \times L^2(\Omega) \times L^2_\sigma(\Omega) \times (L^2(\Omega))^{2\times 2}\right]$$

there exists a global weak solution of the regular problem with

$$\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \qquad \frac{\partial \phi}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$$

$$\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega)), \qquad \frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;H^{-1}_{0,\sigma}(\Omega))$$

$$\mathbf{C}, q \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \qquad \frac{\partial q}{\partial t}, \frac{\partial \mathbf{C}}{\partial t} \in L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))$$

$$\mu \in L^2(0,T;H^1(\Omega)),$$

which obey the energy inequality

Existence of global weak dissipative solutions

1. Existence of solutions 2D ✓
2. Main problem

$$\frac{1}{2}\left[\nabla \mathbf{u}\mathbf{C} + \mathbf{C}(\nabla \mathbf{u})^T\right] : \mathbf{C} \neq \mathrm{tr}(\mathbf{C})\mathbf{C} : \nabla \mathbf{u} \text{ in 3D}$$

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4. We only have

$$\frac{1}{2}\mathrm{tr}(\mathbf{C})\mathrm{tr}\left[\nabla \mathbf{u}\mathbf{C} + \mathbf{C}(\nabla \mathbf{u})^T\right] = \mathrm{tr}(\mathbf{C})\mathbf{C} : \nabla \mathbf{u}$$

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5. Existence, regularity & weak-strong uniqueness for the Peterlin model in 3D
   ... Brunk, M.L., Lu: Commun. Math. Sci. 2022
6. Extension to a reduced viscoelastic phase separation model in 3D
   ... Brunk: DCDS 2022

Relative energy

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Let $E(z)$ be a convex energy, then the relative energy is given by

$$\mathcal{E}(z|\hat{z}) := E(z) - E(\hat{z}) - \left\langle \frac{\delta E}{\delta z}(\hat{z}), z - \hat{z} \right\rangle$$

- for non-convex double-well potential add a penalty term
- f is λ -convex if $f(z) + \lambda |z|^2$ is convex for some $\lambda > 0$
- for $d = 2$

$$\mathcal{E}(z|\hat{z}) := \int_\Omega \frac{\gamma}{2} |\nabla(\phi - \hat{\phi})|^2 + f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi - \hat{\phi}) + \frac{\alpha}{2}|\phi - \hat{\phi}|^2 + \frac{1}{2}|q - \hat{q}|^2 + \frac{1}{2}|\mathbf{u} - \hat{\mathbf{u}}|^2 + \frac{1}{4}|\mathbf{C} - \hat{\mathbf{C}}|^2$$

$\alpha > \lambda := \max\{\gamma, \gamma - \min f''\}$

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1. Uniqueness of solution $\longrightarrow$ *weak-strong uniqueness*
2. Solution depends continuously on data $\longrightarrow$ *stability estimate*

▶ Weak-strong uniqueness [Brunk, ML (ZAMM 2022)]

Compare a weak solution z with a more regular (strong) solution $\hat{z}$ starting from the same initial data

$$\mathcal{E}(z|\hat{z})(t) + \mathcal{D}(t) \leq c(t)\mathcal{E}(z|\hat{z})(0) = 0$$

$$\begin{aligned}\mathcal{D} = \tfrac{1}{2}\int_0^t \int_\Omega & |n(\phi)\nabla(\mu - \hat{\mu}) - \nabla(A(\phi)(q - \hat{q}))|^2 + \frac{1}{\tau(\phi)}|q - \hat{q}|^2 \\ & + \varepsilon_1 |\nabla(q - \hat{q})|^2 + \eta(\phi)|\mathrm{D}\mathbf{v} - \mathrm{D}\hat{\mathbf{v}}|^2 + \frac{\varepsilon_2}{2}|\nabla(\mathbf{C} - \hat{\mathbf{C}})|^2 \\ & + h(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C})|\mathbf{C} - \hat{\mathbf{C}}|^2 \mathrm{d}x\, \mathrm{d}t'\end{aligned}$$

$\longrightarrow$ $z = \hat{z}$ weak-strong uniqueness

Stability estimate

Compare a weak solution z with arbitrary functions $\hat{z}$:

$$\mathcal{E}(z|\hat{z})(t) + \mathcal{D}(t) \le e^{c(t)}\mathcal{E}(z|\hat{z})(0) + Ce^{c(t)} \int_0^t \sum_{i=1}^{5} \|r_i\|_*^2.$$

- residuals r_i study parameter changes, asymptotic limits, coarse-graining errors for a hierarchy of models (relative entropy) and convergence of numerical schemes

Error estimates I

- rigorous error estimate for the Cahn-Hilliard equation through the relative energy:
  Brunk, Egger, Habrich, ML: M²AN'23

$$\mathcal{E}(z|\hat{z}) := \int \frac{\gamma}{2}|\nabla(\phi-\hat{\phi})|^2 + f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi-\hat{\phi}) + \frac{\alpha}{2}|\phi-\hat{\phi}|^2$$

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- periodic BC, second order conforming finite elements
- Cranck-Nicolson-type in time
- Let (ϕ, μ) be a regular periodic weak solution with $\phi_0 \in H^3(\Omega)$ s.t.

$$\phi \in H^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^3(\Omega)),$$
$$\mu \in H^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,3}(\Omega)),$$

  and let $(\phi_{h,\tau}, \mathbf{u}_{h,\tau})$ be a solution of FEM.
- Then

$$\max_{t^n} \|\phi_{h,\tau}(t^n) - \phi(t^n)\|_{H^1} + \left\|\overline{\mu}_{h,\tau} - \bar{\mu}\right\|_{L^2(0,T;H^1)} \leq C_T(h^2 + \tau^2)$$

Error estimates II

- error estimates for Navier-Stokes-Cahn-Hilliard system:

Brunk, Egger, Habrich, ML: M^3AS 2023

- Taylor-Hood (P2/P1) finite elements for the velocity and pressure
- P2 finite elements for ϕ, μ
- two-, three-dimensional model, periodic BC

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Let $(\phi, \mu, \mathbf{u}, p)$ is a regular sol., s.t.

$$\phi \in H^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^3(\Omega)), \quad \mu \in H^2(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)$$
$$\mathbf{u} \in H^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^3(\Omega)), \quad p \in L^2(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega)$$

Then

$$\|\phi_{h,\tau} - \phi\|_{L^\infty_t, H^1_x} + \|\overline{\mu}_{h,\tau} - \bar{\mu}\|_{L^2_t, H^1_x} +$$
$$\|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty_t, L^2_x} + \|\overline{\mathbf{u}}_{h,\tau} - \overline{\mathbf{u}}\|_{L^\infty_t, L^2_x} \leq C(T)(h^2 + \tau^2)$$

$\overline{\mathbf{u}} \ldots$ p.w. constant projection in time on (t^n, t^{n+1})

Error estimates III

- error estimates for Cahn-Hilliard model with bulk stress q:
Brunk, Egger, Habrich, ML: ArXiv 2024
- P2 finite elements for ϕ, μ, q
- two-, three-dimensional model, periodic BC
  - Let (ϕ, μ, q) is a regular sol., s.t.

$$\phi \in H^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^3(\Omega)),$$
$$\mu \in H^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,3}(\Omega)) \cap L^2(0,T;H^3(\Omega))$$
$$q \in H^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,3}(\Omega)) \cap L^2(0,T;H^3(\Omega))$$

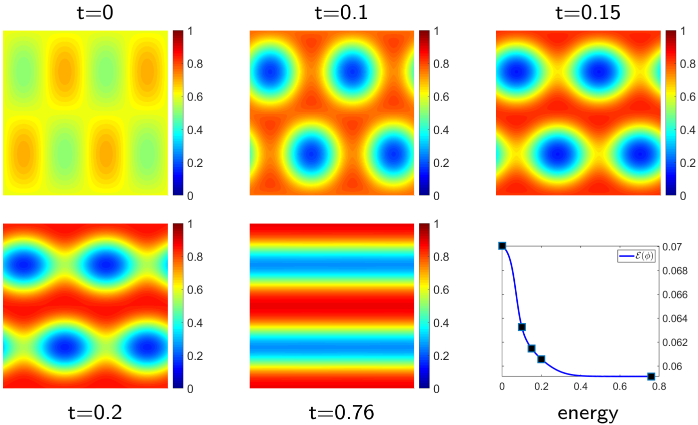
$$\|\phi_{h,\tau} - \phi\|_{L^\infty_t, H^1_x} + \|q_{h,\tau} - q\|_{L^\infty_t, L^2_x} + \|\overline{\mu}_{h,\tau} - \bar{\mu}\|_{L^2_t, H^1_x} + \|\overline{q}_{h,\tau} - \overline{q}\|_{L^2_t, H^1_x} \le C(T)(h^2 + \tau^2)$$

$\overline{q}$ ... p.w. constant projection in time on (t^n, t^{n+1})

Cahn-Hilliard eq.

- potential $f(\phi) = \frac{3}{2}(\phi \log(\phi) + (1-\phi)\log(1-\phi) + 3\phi(1-\phi))$

$$\phi_0(x,y) = 0.1 \sin(4\pi x) \sin(2\pi y) + 0.6$$



Convergence rates

Table: Errors and convergence rates for the semi-discrete and fully-discrete approximations

k	e_h	eoc	$e_{h,\tau}$	eoc
0	$5.9075 \cdot 10^{-1}$	—	$5.9426 \cdot 10^{-1}$	—
1	$1.7016 \cdot 10^{-1}$	1.80	$1.7109 \cdot 10^{-1}$	1.79
2	$4.3761 \cdot 10^{-2}$	1.96	$4.4155 \cdot 10^{-2}$	1.95
3	$1.1038 \cdot 10^{-2}$	1.99	$1.1251 \cdot 10^{-2}$	1.97
4	$2.7658 \cdot 10^{-3}$	2.00	$3.3468 \cdot 10^{-3}$	1.74

Viscoelastic separation

- Cahn-Hilliard and q-equation

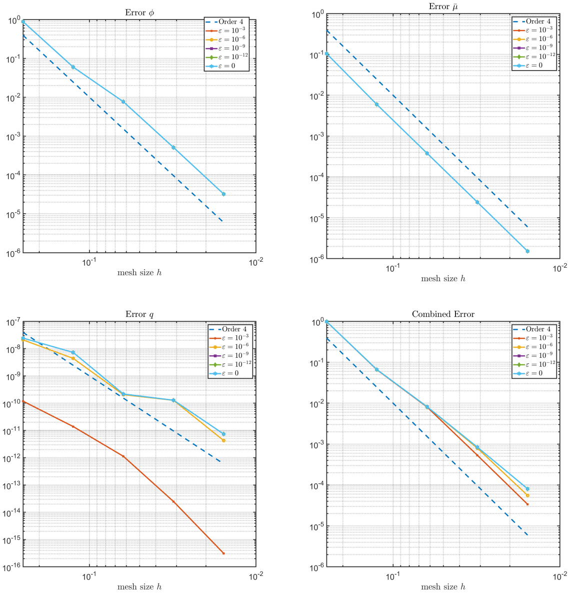
initial data:

$$\phi_0(x,y) = 0.25\cos(2\pi x)\cos(2\pi y) + 0.5, \quad q_0(x,y) = 0.01\sin(2\pi x)\sin(2\pi y).$$

$\gamma = 10^{-3}$
$M(\phi) = [\frac{4}{\sqrt{10}} \cdot \phi(1-\phi)]^2 + \varepsilon$, $f(\phi) = (\phi - 0.95)^2(\phi - 0.05)^2$,
$\tau_b(\phi) = 10^2(10\phi^2 + 10^{-4})$, and

$$A(\phi) = 5 \cdot 10^{-3} \cdot \left[1 + \tanh(5[\cot(\pi\phi^*) - \cot(\pi\phi)])\right]$$

with $\phi^* = \int_\Omega \phi_0$ denoting the total mass



Convergence rates for the errors in ϕ (top left), $\bar{\mu}$ (top right), q (middle left), $\bar{q}$ (middle right) and the combined error (bottom) for different ε

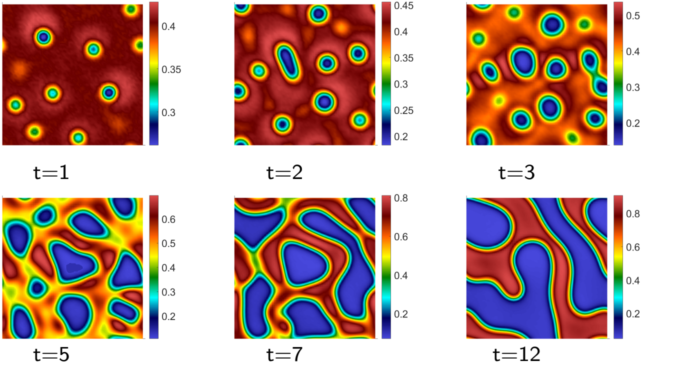
- Cahn-Hilliard and q-equation

initial data:

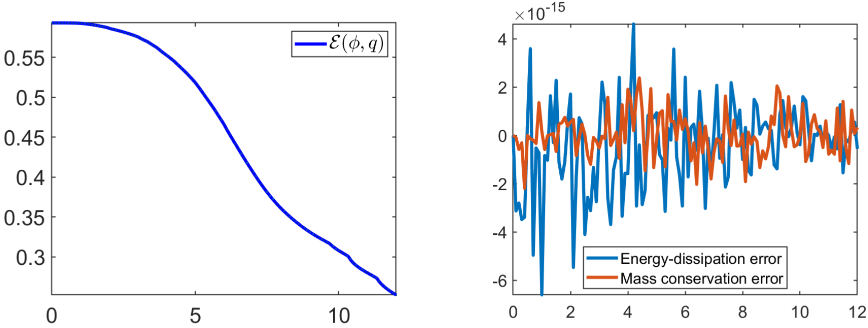
$\phi_0(x,y) = 0.4 + \xi(x,y), \quad q_0 = 0, \quad \xi(x,y) \sim \mathcal{U}[-0.0025, 0.0025]$

$\gamma = \varepsilon = 10^{-3}, \;\; f(\phi) = (\phi - 0.95)^2(\phi - 0.05)^2$

$$A(\phi) = \frac{1}{2}\big[1 + \tanh(10[\cot(\pi\phi^*) - \cot(\pi\phi)])\big], \quad \phi^* = \int_\Omega \phi_0$$



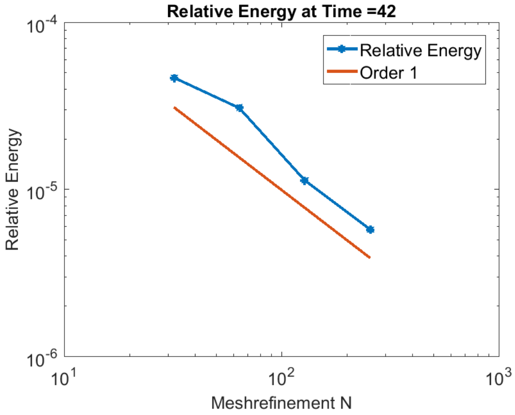
Snapshots of the volume fraction ϕ with pure phases at $\phi = 0$ and $\phi = 1$



Temporal evolution of the energy (left) and the conservation and energy-dissipation error (right)

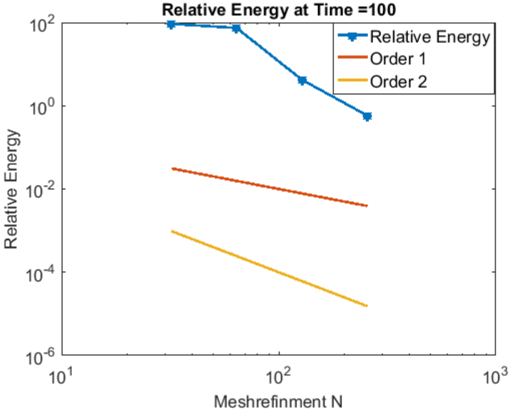
Full viscoelastic separation

- characteristic FEM
  - $\phi_0(x) = 0.4 + \xi(x), q_0 = 0, \mathbf{u}_0 = \mathbf{0}, \mathbf{C}_0 = \frac{1}{\sqrt{2}}\mathbf{I}$
    $\xi(x)$ from $[-10^{-3}, 10^{-3}]$



Full viscoelastic separation

- $\phi_0(x) = 0.4 + 0.2 \sin\left(\frac{2\pi x}{128}\right) \sin\left(\frac{2\pi y}{128}\right), q_0 = 0, \mathbf{u}_0 = \mathbf{0}, \mathbf{C}_0 = \frac{1}{\sqrt{2}}\mathbf{I}$



Peterlin viscoelastic system in 3D

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\big(\eta \mathrm{D}\mathbf{u}\big) - \nabla p + \operatorname{div}\big(\mathbf{T}\big),\\
\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C} &= (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top\\
&\quad + \Phi(\operatorname{tr}(\mathbf{C}))\mathbf{I} - \chi(\operatorname{tr}(\mathbf{C}))\mathbf{C} + \varepsilon\Delta\mathbf{C},\\
\operatorname{div}\big(\mathbf{u}\big) &= 0, \quad \mathbf{T} = \operatorname{tr}(\mathbf{C})\mathbf{C}.
\end{aligned}
$$

- $\Phi := \operatorname{tr}(\mathbf{C}) + a$ and $\chi := \operatorname{tr}(\mathbf{C})^2 + a|\operatorname{tr}(\mathbf{C})|$ for a given $a \geq 0$

Weak dissipative sol. in 3D / Peterlin model

Theorem (Brunk, ML, Lu (DCDS'22))

• *initial data* $(\mathbf{u}_0, \mathbf{C}_0) \in [L^2_\sigma \times L^2(\Omega)^{3\times 3}_{SPD}]$, $T > 0$.
There exists a global weak solution of the Peterlin system, s.t.

$$\mathbf{u} \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T;V) \cap C([0,T]; L^q(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;V^*),$$

$$\mathbf{C} \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap L^4(\Omega_T)$$

$$\cap\, C([0,T]; L^q(\Omega)) \cap W^{1,\frac{4}{3}}(0,T; H^{-1}(\Omega)),$$

$$\chi(\mathrm{tr}(\mathbf{C}))\mathbf{C} \in L^{\frac{4}{3}}(\Omega_T), \qquad \Phi(\mathrm{tr}(\mathbf{C})) \in L^2(\Omega_T), \quad \textit{for any } 1 \le q < 2$$

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$$\cap\, C([0,T]; L^q(\Omega)) \cap W^{1,\frac{4}{3}}(0,T; H^{-1}(\Omega)),$$

$$\chi(\mathrm{tr}(\mathbf{C}))\mathbf{C} \in L^{\frac{4}{3}}(\Omega_T), \qquad \Phi(\mathrm{tr}(\mathbf{C})) \in L^2(\Omega_T), \quad \textit{for any } 1 \le q < 2$$

• *the energy inequality holds a.e.* $t \in (0,T)$

$$\left(\int_\Omega \frac{1}{2}|\mathbf{u}(t)|^2 + \frac{1}{4}|\mathrm{tr}(\mathbf{C}(t))|^2 \,\mathrm{dx}\right) + \int_0^t \int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon}{2}|\nabla \mathrm{tr}(\mathbf{C})|^2 \,\mathrm{dx}\,\mathrm{d}\tau +$$
$$\int_0^t \int_\Omega \frac{1}{2}|\mathrm{tr}(\mathbf{C})|^4 + \frac{a}{2}|\mathrm{tr}(\mathbf{C})|^3 \,\mathrm{dx}\,\mathrm{d}\tau$$
$$\le \int_0^t \int_\Omega \frac{1}{2}|\mathrm{tr}(\mathbf{C})|^2 + \frac{a}{2}\mathrm{tr}(\mathbf{C}) \,\mathrm{dx}\,\mathrm{d}\tau + \left(\int_\Omega \frac{1}{2}|\mathbf{u}(0)|^2 + \frac{1}{4}|\mathrm{tr}(\mathbf{C}(0))|^2\right)$$

- If $a = 0$ the conformation tensor $\mathbf{C}$ is symmetric positive semi-definite
- If $a > 0$ and $\mathrm{tr}(\log \mathbf{C}_0) \in L^1(\Omega)$ then $\mathbf{C}$ is symmetric positive definite and

$$\begin{aligned}
&\mathrm{tr}(\log \mathbf{C}) \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&\mathrm{tr}(\mathbf{C}^{-1}), \mathrm{tr}(\mathbf{C}^{-1})\mathrm{tr}(\mathbf{C}) \in L^1(0,T;L^1(\Omega)).
\end{aligned}$$

$$\begin{aligned}
&\left(\int_\Omega \frac{1}{2}|\mathbf{u}(t)|^2 + \frac{1}{4}|\mathrm{tr}(\mathbf{C}(t))|^2 - \frac{1}{2}\mathrm{tr}(\log \mathbf{C}(t))\,\mathrm{dx} \right)\\
&+ \int_0^t \int_\Omega \eta |\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon}{2}|\nabla \mathrm{tr}(\mathbf{C})|^2 + \frac{\varepsilon}{6}|\nabla \mathrm{tr}\,(\log \mathbf{C})|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I})\,\mathrm{dx}\,\mathrm{d}\tau\\
&\leq \left(\int_\Omega \frac{1}{2}|\mathbf{u}(0)|^2 + \frac{1}{4}|\mathrm{tr}(\mathbf{C}(0))|^2 - \frac{1}{2}\mathrm{tr}(\log \mathbf{C}(0))\,\mathrm{dx} \right).
\end{aligned}$$

Hybrid kinetic-macroscopic model

- Fokker-Planck equation for the viscoelastic part

$$\partial_t \phi + \mathbf{u} \cdot \nabla_{\mathbf{x}} \phi = \mathrm{div}_{\mathbf{x}} \Big((1+\varepsilon_0) m(\phi) \nabla_{\mathbf{x}} \mu - m^{1/2}(\phi) \nabla_{\mathbf{x}} (A(\phi) q) \Big) ,$$

$$\mu = -\gamma \Delta_{\mathbf{x}} \phi + f'(\phi),$$

$$\partial_t q + \mathbf{u} \cdot \nabla_{\mathbf{x}} q = -\frac{1}{\tau(\phi)} q + \varepsilon_1 \Delta_{\mathbf{x}} q + A(\phi) \mathrm{div}_{\mathbf{x}} \Big(\nabla_{\mathbf{x}} (A(\phi) q) - m^{1/2}(\phi) \nabla_{\mathbf{x}} \mu \Big) ,$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} = \mathrm{div}_{\mathbf{x}} (\eta(\phi) \mathrm{D}_{\mathbf{x}} \mathbf{u} - p\mathbf{I} + \mathbf{T}(\psi)) - \phi \nabla_{\mathbf{x}} \mu,$$

$$\mathrm{div}_{\mathbf{x}} (\mathbf{u}) = 0,$$

$$\mathbf{T}(\psi) = \gamma_3(\langle |\mathbf{R}|^2 \rangle) \langle \mathbf{R} \otimes \mathbf{R} \rangle,$$

$$\partial_t \psi + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi + \mathrm{div}_{\mathbf{R}} (\nabla_{\mathbf{x}} \mathbf{u} \mathbf{R} \psi)$$
$$= \gamma_2(\phi, \langle |\mathbf{R}|^2 \rangle) \Delta_{\mathbf{R}} \psi + \mathrm{div}_{\mathbf{R}} \Big(\gamma_1(\phi, \langle |\mathbf{R}|^2 \rangle) \mathbf{R} \psi \Big) + \varepsilon_2 \Delta_{\mathbf{x}} \psi$$

$\gamma_1, \gamma_2, \gamma_3$ are connected to the functions χ, Φ in the Peterlin model

$\langle f \rangle \equiv \int_{R^3} f(\mathbf{R}) \psi(\mathbf{x}, t, \mathbf{R}) \mathrm{d}\mathbf{R} \ \ldots \ \psi$ is the probability density distribution
$\mathbf{C}(\psi) \equiv \langle \mathbf{R} \otimes \mathbf{R} \rangle$

Existence of weak solutions for the kinetic system

▶ Existence of weak solutions for the kinetic system

Let $d = 2$ and assume that $\gamma_1(s) = s^2, \gamma_2(s) = s, \gamma_3(s) = s$.

1. Then there exists at least one global weak solution of the hybrid kinetic-macroscopic viscoelastic phase separation model.
2. Rigorous macroscopic closure: the macroscopic viscoelastic phase separation model (Cahn-Hilliard-Navier-Stokes-Peterlin) is the closure of the hybrid kinetic-macrosopic model
3. $\mathbf{C} = \langle \mathbf{R} \otimes \mathbf{R} \rangle$