An update on the analysis of the multiphase compressible flows

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Mixtures: Modeling, analysis and computing, Prague, 5-7 February 2025

Compressible mixture

Constant temperature, non-reactive, two-component gas:

$$\partial_t \rho_k + \operatorname{div}(\rho_k \mathrm{u}) + \operatorname{div} \mathrm{F}_k = 0, \quad k = 1, 2,$$
$$\partial_t(\rho \mathrm{u}) + \operatorname{div}(\rho \mathrm{u} \otimes \mathrm{u}) - \operatorname{div} \mathrm{S} + \nabla p = 0,$$
$$\rho = \rho_1 + \rho_2.$$

- The unknowns: u and ρ_k, $k = 1, 2$.
- The pressure: $p = \sum_{k=1}^{n} \frac{\rho_k}{m_k}$.
- The stress tensor: $\mathrm{S} = 2\mu \mathrm{D}(\mathrm{u}) + \nu \operatorname{div} \mathrm{u} \mathrm{I}$, $\mu > 0$, $2\mu + 3\nu \geq 0$.

Giovangigli, Bothe, Jüngel, Stelzer, Marion, Temam, Dryer, Druet, Gajewki, Guhlke, Bulicek, Pokorný, Zamponi, Piasecki, Shibata, Z.

Compressible two-phase flows

Two immiscible compressible fluids sharing the same velocity:

$$
\begin{aligned}
&\partial_t(\alpha_k\rho_k) + \operatorname{div}(\alpha_k\rho_k \mathrm{u}) = 0, \quad k = 1,2,\\
&\partial_t(\rho \mathrm{u}) + \operatorname{div}(\rho \mathrm{u}\otimes \mathrm{u}) - \operatorname{div} \mathrm{S} + \nabla p = 0,\\
&\rho = \alpha_1\rho_1 + \alpha_2\rho_2,\\
&\alpha_1 + \alpha_2 = 1.
\end{aligned}
$$

- The unknowns: α_k and ρ_k, $k = 1, 2$.
- The pressure: $p = p_1(\rho_1) = p_2(\rho_2)$.
- The stress tensor: $\mathrm{S} = 2\mu \mathrm{D}(\mathrm{u}) + \nu \operatorname{div} \mathrm{u} \mathrm{I}$, $\mu > 0$, $2\mu + 3\nu \geq 0$.

M. Ishii, T Hibiki; Bresch, Desjardins, Ghidaglia, Grenier, Hilliairet, Novotný, Jin, Pokorný, Mucha, Sun, Li., Z.,....

Mixtures vs multi-phase flows

Similarities:

- Multiple continuity equations.
- Pressure depending on more than one densities.
- The same form of the stress tensor.

Differences:

- One more unknown in the multi-phase model.
- One density in the mixture model: $\rho_k = Y_k \rho$, where $\sum_k Y_k = 1$, vs multiple densities in the multi-phase model.
- The diffusion fluxes (mixtures are mixing!)
- Different forms of the pressure.

Two velocities two-phase model

The model of two-phase flow of compressible fluids:

$$\begin{cases} \partial_t(\alpha^\pm \rho^\pm) + \operatorname{div}(\alpha^\pm \rho^\pm \mathrm{u}^\pm) = 0, \\ \partial_t(\alpha^\pm \rho^\pm \mathrm{u}^\pm) + \operatorname{div}(\alpha^\pm \rho^\pm \mathrm{u}^\pm \otimes \mathrm{u}^\pm) + \alpha^\pm \nabla p^\pm - \operatorname{div} \mathrm{S}(\alpha^\pm \mathrm{u}\pm) = M^\pm, \\ \alpha^+ + \alpha^- = 1, \\ p^+ = (\rho^+)^{\gamma^+} = (\rho^-)^{\gamma^-} = p^- := p. \end{cases}$$

- The unknowns: α^+, α^- – volume fractions, ρ^+, ρ^- – species' densities, u^+, u^- – species velocities.
- The relaxation term $M^\pm = D_\mathrm{u}(\mathrm{u}_\mp - \mathrm{u}\pm)$.
- The possible closure assumptions:
  -M. Ishii '75, M. Ishii, T. Hibiki '06 (saturation condition $p^+ = p^-$).
  -M.R. Baer, J W. Nunziato '86 ($\partial_t \alpha^+ + \mathrm{u}_{int} \cdot \nabla \alpha^+ = D_p(p^+ - p^-)$).
  -Bresch, Desjardins, Ghidaglia, Grenier, Hillairet '18.

Gas-liquid asymptotic model in 1D

Once upon a time, a PhD student told me a story...

- Let $\rho^+ << \rho^- = const.$, ρ_0 average of ρ^+, $\tilde{\rho} = \frac{\rho^+}{\rho_0}$ and $\varepsilon = \frac{\rho_0}{\rho^-}$.
- Let $u^+ = u^- + \varepsilon w$.
- Let $p = (\rho^+)^\gamma$, under the conjecture that $\varepsilon \rho_0^{\gamma-1} p \to \pi$ with $(1-\alpha^-)\pi = 0$.

Then the equations for "$-$"phase lead to congested system:

$$\begin{cases} \partial_t(\alpha^-) + \partial_x(\alpha^- \mathrm{u}^-) = 0, \\ \partial_t(\alpha^- \mathrm{u}^-) + \partial_x(\alpha^- \mathrm{u}^- \mathrm{u}^-) + \partial_x \pi = 0, \\ 0 \le \alpha^- \le 1, \quad (1-\alpha^-)\pi = 0, \end{cases}$$

Bouchut, Brenier, Cortes, Ripoll '01.

Bresch, Perrin, Z. '15, and subsequent results with: Degond, Vauchelet, Navoret and Chaudhuri, Mehmood, Lefebvre-Lepot...

Analysis of two-phase gas-gas fluid model

Reduction of the model

- barotropic pressure law $p^+ = (\rho^+)^{\gamma^+}$, $p^- = (\rho^-)^{\gamma^-}$,
- saturation condition $p^+ = p^-$,
- the relaxation parameter $D_{\mathrm{u}} \to \infty$, i.e. $\mathrm{u}^+ = \mathrm{u}^- = \mathrm{u}$.

So, denoting

$$R = \alpha^+\rho^+, \quad Q = \alpha^-\rho^-, \quad \rho = R + Q,$$

the system reduces to:

$$\begin{aligned}
&\partial_t(R) + \mathrm{div}(R\mathrm{u}) = 0,\\
&\partial_t(Q) + \mathrm{div}(Q\mathrm{u}) = 0,\\
&\partial_t(\rho\mathrm{u}) + \mathrm{div}(\rho\mathrm{u} \otimes \mathrm{u}) - \mu\Delta\mathrm{u} - (\lambda + \mu)\nabla\,\mathrm{div}\,\mathrm{u} + \nabla p(R, Q) = 0.
\end{aligned}$$

A. Vasseur, H. Wen, C. Yu '17, Bresch, Mucha, Zatorska '19 Novotný, M. Pokorný '20.

When $\alpha^- = 1$

Homogeneous compressible, viscous, isothermal fluid equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla p(\rho) = 0. \end{cases}$$

- Pressure $p(\rho)$ is givent function of the density, usually, ρ^γ, $\gamma > 1$.
- Fixed viscosity coefficients: $\mu > 0$ and $\lambda \geq -\frac{2}{3}\mu$.
- Dirichlet b.c., whole space case, periodic domain...

Lions '98 ($\gamma \geq \frac{9}{5}$); Feireisl '01 ($\gamma > \frac{5}{3}$); Bresch, Jabin '18 (p non-monotone), Chaudhuri, Mucha, Zatorska '23 (complete construction using convolutions).

Other results for two-component flows

- Existence and stability of unique weak solutions in 1D:

$$\left(\|R-\widetilde{R}\|_{L^\infty(\Omega_T)}+\|Q-\widetilde{Q}\|_{L^\infty(\Omega_T)}+\|u-\widetilde{u}\|_{V_2(\Omega_T)}\right)$$
$$\leq C\left(\|R_0-\widetilde{R_0}\|_{L^\infty}+\|Q_0-\widetilde{Q_0}\|_{L^\infty}+\|u_0-\widetilde{u_0}\|_{L^2}\right),$$

  and the exponential decay to the unique steady state

$$\|(R-R_\infty, Q-Q_\infty, u-u_\infty)\|_{L^2}\leq C_1\exp(-C_2 t).$$

  Li, Sun, Z. 2020, Zlotnik 1992.

- Nonuniqueness of (weak) solutions in higher dimensions (inviscid case)

  Li, Zatorska 2021, De Lellis, Székelyhidi Jr. 2009.

- Short-time existence and maximal regularity of solutions

  Piasecki, Zatorska 2022, Enomoto, Shibata 2013, Piasecki, Shibata Zatorska 2019, 2020.

Weak-strong uniqueness of solutions

We consider the compressible two-fluid model on $\mathbb{T}^3$:

$$\begin{cases} \partial_t R + \mathrm{div}_x(R\mathrm{u}) = 0, \\ \partial_t Q + \mathrm{div}_x(Q\mathrm{u}) = 0, \\ \partial_t[(R+Q)\mathrm{u}] + \mathrm{div}[(R+Q)\mathrm{u}\otimes\mathrm{u}] + \nabla p(Z) = \mu\Delta\mathrm{u} + (\mu+\lambda)\nabla\,\mathrm{div}\,\mathrm{u}. \end{cases}$$

Theorem (Li, Z.'22)

Let (R, Q, u) *be a finite energy weak solutions to such that*

$$(R, Q) \in L^\infty(0, T; L^\infty(\mathbb{T}^3)).$$

Assume that $(\tilde{R}, \tilde{Q}, \tilde{\mathrm{u}})$ *is the classical solution to the same problem on* $[0, T]$, *starting from the same initial data. Then*

$$R = \tilde{R},\ \ Q = \tilde{Q},\ \ \mathrm{u} = \tilde{\mathrm{u}} \quad \textit{in } [0, T] \times \mathbb{T}^3.$$

P. Germain: Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. J. Math. Fluid Mech. 13, 137–146 (2011).

Unconditional result missing

The energy functional for the real bi-fluid system:

$$\int_{\mathbb{T}^d} \frac{1}{2}(R+Q)|\mathrm{u}|^2 + \underbrace{\frac{1}{\gamma^+ - 1}\left(\frac{R}{\alpha}\right)^{\gamma^+}\alpha + \frac{1}{\gamma^- - 1}\left(\frac{Q}{1-\alpha}\right)^{\gamma^-}(1-\alpha)}_{:=H(\mathcal{R},\mathcal{Q})=\frac{1}{\gamma^+-1}\mathcal{R}^{\gamma^+}+\frac{1}{\gamma^--1}\mathcal{Q}^{\gamma^-}} \,\mathrm{d}x$$

for

$$\mathcal{R} = \alpha^{1/\gamma^+ - 1} R, \quad \mathcal{Q} = (1-\alpha)^{1/\gamma^- - 1} Q.$$

The natural candidate for the relative entropy functional would be:

$$\begin{aligned} &\mathcal{E}(R, Q, \mathrm{u}|\bar{R}, \bar{Q}, \mathrm{v})(t) \\ &= \int_{\mathbb{T}^d} \left(\frac{1}{2}(R+Q)|\mathrm{u}-\mathrm{v}|^2\right) \mathrm{d}x \\ &\quad + \int_{\mathbb{T}^d} \left(H(\mathcal{R},\mathcal{Q}) - H_{\mathcal{R}}(\bar{\mathcal{R}},\bar{\mathcal{Q}})(\mathcal{R}-\bar{\mathcal{R}}) - H_{\mathcal{Q}}(\bar{\mathcal{R}},\bar{\mathcal{Q}})(\mathcal{Q}-\bar{\mathcal{Q}}) - H(\bar{\mathcal{R}},\bar{\mathcal{Q}})\right) \mathrm{d}x \end{aligned}$$

but the pressure potential is <u>not convex</u> w.r.t. conservative variables!

Jin, Kwon, Necasova and Novotný '21 (under artificial convexity assumption).

Jin, Novotn'y '19 (for Baer-Nunziato system).

Pressureless (high Mach number) limit

We consider $\delta \to 0^+$ in the system

$$\begin{cases} \partial_t r^\pm + \partial_x \left(r^\pm u\right) = 0, \\ \partial_t \left(r^\pm u\right) + \partial_x \left(r^\pm u^2\right) + \delta \alpha^\pm \partial_x p^\pm - \nu \partial_x \left(r^\pm \partial_x u\right) = 0, \\ \alpha^+ + \alpha^- = 1, \\ p^+ = p^-, \end{cases}$$

where $r^\pm = \alpha^\pm \rho^\pm$.

Bresch, Huang Li, 2012.

The new (κ-)relative entropy is based on the "two-velocity" formulation using

$$w^\pm = \nu \sqrt{\kappa\left(1-\kappa\right)} \partial_x \log r^\pm, \quad v^\pm = u + \nu\kappa\partial_x \log r^\pm = u + \sqrt{\frac{\kappa}{1-\kappa}} w^\pm.$$

Equation for $\rho^\pm w^\pm$ is more or less parabolic, $\rho^\pm v^\pm$ is hyperbolic.

Relative entropy

Note that there is no transport equation for α^+ (equivalently α^-); we have:

$$\partial_t \alpha^+ + u\partial_x \alpha^+ + \frac{\frac{\gamma^+}{\gamma^-} - 1}{\frac{\gamma^+}{\gamma^-}\alpha^- + \alpha^+}\alpha^+\alpha^-\partial_x u = 0.$$

Treating α's as fixed parameters (s.t. $\alpha^+ - \alpha^-$), we define:

$$\mathcal{H}_\alpha\left(\rho^+, \rho^-\right) := \frac{1}{\gamma^+ - 1}\alpha^+\left(\rho^+\right)^{\gamma^+} + \frac{1}{\gamma^- - 1}\alpha^-\left(\rho^-\right)^{\gamma^-},$$

and then our relative entropy reads:

$$\begin{aligned}
&\tilde{\mathcal{E}}\left(r^\pm, v^\pm, w^\pm | \tilde{r}^\pm, \tilde{v}^\pm, \tilde{w}^\pm\right) \\
&= \frac{1}{2}\int_{\mathbb{T}} \left(r^+\left(v^+ - \tilde{v}^+\right)^2 + r^+\left(w^+ - \tilde{w}^+\right)^2 + r^-\left(v^- - \tilde{v}^-\right)^2 + r^-\left(w^- - \tilde{w}^-\right)^2\right) dx \\
&+ \delta\int_{\mathbb{T}} \Big[\mathcal{H}_\alpha\left(\rho^+, \rho^-\right) - \partial_{\rho^+}\mathcal{H}_\alpha\left(\tilde{\rho}^+, \tilde{\rho}^-\right)\left(\rho^+ - \tilde{\rho}^+\right) \\
&\qquad - \partial_{\rho^-}\mathcal{H}_\alpha\left(\tilde{\rho}^+, \tilde{\rho}^-\right)\left(\rho^- - \tilde{\rho}^-\right) - \mathcal{H}_\alpha\left(\tilde{\rho}^+, \tilde{\rho}^-\right)\Big] dx.
\end{aligned}$$

Chiarello, Donatelli, Parmeshwar, Zatorska, 2025.

Thank you !

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