Recent trends in mathematical modelling

Workshop, 30th November – 4th December 2022, Kácov

Schedule

Thursday, December 1

Time	Speaker	Title
8:00		Breakfast
8:50		Opening
9:00	Josef Málek	On flows of viscoelastic rate type fluids with temperature dependent material coefficients
9:40	Petr Kaplický	Uniqueness and regularity of flows of non-Newtonian fluids
10:20		Coffee & Refreshment
	Chair: Petr Kaplický	
10:40	Ondřej Kreml	Wild solutions to isentropic Euler equations starting from smooth initial data
11:20	Jakub Woźnicki	Non-Newtonian fluids with discontinuous-in-time stress tensor
12:00		Lunch
	Chair: Tabea Tscherpel	
14:30	Anna Abbatiello	On the decay in time of solutions to a generalized Navier-Stokes-Fourier system
15:10	Dalibor Pražák	On linearization principle for the generalized NSE with dynamic boundary condition
15:50	Erika Maringová	On implicit constitutive relations
16:30		Coffee & Refreshment
	Chair: Dalibor Pražák	
16:50	Daniel Lear	Traveling waves close to the Couette flow
17:30	Buddhika Priyasad	Finite-dimensional uniform stabilization of the Boussinesq system in Besov spaces
19:00		Dinner

Friday, December 2

Time	Speaker	Title
8:00		Breakfast
	Chair: Alexis Vasseur	
9:00	Lars Diening	Kac(an)ov iteration
9:40	Milan Pokorný	Homogenization of Navier–Stokes–Fourier system in domains with tiny holes
10:20		Coffee & Refreshment
	Chair: Lars Diening	
10:40	Andrea Cianchi	Optimal embeddings for symmetric gradient Orlicz-Sobolev spaces
11:20	Václav Mácha	Local-in-time existence of strong solutions to a class of the compressible non-Newtonian Navier-Stokes equations
12:00		Lunch
	Chair: Mark Steinhauer	
14:30	Alexis Vasseur	Boundary vorticity estimate for the Navier-Stokes equation and control of layer separation in the inviscid limit
15:10	Ondřej Souček	Vapor plume on Enceladus – a modelling journey there and back again
15:50	Martin Kružík	Nonlinear and linearized models in thermoviscoelasticity
16:30		Coffee & Refreshment
	Chair: Martin Kružík	
16:50	Jan Blechta	Preconditioning of H^1-regularized parameter identification
17:30	Piotr Gwiazda	On a range of exponents for absence of Lavrentiev phenomenon for double phase functionals
18:10	Tomáš Bárta	Stokes problem with dynamic boundary conditions
18:30	Michael Zelina	On the uniqueness of the solution and finite-dimensional attractors for the 3D flow with dynamic slip boundary condition
19:00		Dinner

Saturday, December 3

Time	Speaker	Title
8:00		Breakfast
	Chair: Ondřej Kreml	
9:00	Agnieszka Świerczewska-Gwiazda	The Rayleigh–Benard problem for compressible fluid flows
9:40	Tabea Tscherpel	Sobolev stability of the L^2-projection mapping to finite element spaces
10:20		Coffee & Refreshment
	Chair: Milan Pokorný	
10:40	Michal Pavelka	Multiscale and geometric mechanics and thermodynamics
11:00	Tomáš Los	On unsteady flows of viscoelastic fluids of the Giesekus type
11:20	Vojtěch Kulvait	Coupled algorithms to solve propagation based phase contrast tomography
12:30		Lunch
	Chair: Piotr Gwiazda	
14:00	Vít Průša	A thermodynamic framework for non-isothermal phenomenological models of Mullins effect
14:40	Nicola Fusco	Regularity of capillarity droplets with obstacle
15:20	Endre Süli	Corotational Hookean models of dilute polymeric fluids: existence of global weak solutions
16:00		Coffee & Refreshment
	Chair: Mirko Rokyta	
16:20	Jens Frehse	Josef Málek's work at Bonn 1992–2010
17:00	K. R. Rajagopal	TBA
17:40	Miroslav Bulíček, Mark Steinhauer	Modelling And Large-data Existence Keynotes
18:20		Social Programm

Sunday, December 4

Time	Speaker	Title
8:00		Breakfast
	Chair: Karel Tůma	
9:00	Zdeněk Strakoš	A priori estimates of the spectrum and operator preconditioning
9:40	Martin Lanzendörfer	On shear-thinning fluids in porous media in estimating the pore size distribution
0:00	Jaroslav Hron	Computational study of hemodynamical flows in patient specific geometries
10:20		Coffee & Refreshment
	Chair: Vít Půša	
10:40	Christoph Allolio	Mathematical modeling of biointerfaces from membrane fusion to cell division
11:20	Karel Tůma	Viscoelastic rate-type fluids: thermodynamically compatible derivation, applications and simulations
12:00		Closing
12:15		Lunch
13:15		Departure

Abstracts of main talks

Anna Abbatiello, On the decay in time of solutions to a generalized Navier-Stokes-Fourier system. We consider a flow of non-Newtonian heat conducting incompressible fluid in a bounded domain subjected to the homogeneous Dirichlet boundary condition for the velocity field and the spatially inhomogeneous Dirichlet boundary condition for the temperature. The ultimate goal is to show that the fluid converges to equilibrium as time tends to infinity. However, to justify such result, one needs to deal with very special inequalities and very special test functions, which are typically not admissible on the level of weak solutions. We assume no external body forces. For the power-law like models with the power law index bigger than 11/5 in three dimensions, we identify a class of solutions fulfilling the entropy equality and converging to the equilibria exponentially in a proper metric. In fact, we show the existence of a Lyapunov functional for the problem. Consequently, the steady solution is nonlinearly stable and attracts all suitable weak solutions. This is a joint work with Miroslav Bulíček and Petr Kaplický.

Christoph Allolio, Mathematical modeling of biointerfaces from membrane fusion to cell division. Adequately reproducing essential features of the living organism, such as the time evolution cellular and organelle morphology, can be a difficult modeling challenge. Part of the reason for this is the necessity to reduce complexity of and the lack of parameters for the underlying processes. I present a series of new techniques and intermediate results which aim to relate complex local interactions and active processes to observed phenomena in living cells. Examples include bacterial dvision, mitochondrial fission and endosomal escape. The bridging of scales is performed using continuum theory with input from Cryo-EM and molecular dynamics simulations.

Tomáš Bárta, Stokes problem with dynamic boundary conditions. We study the Stokes problem with dynamic boundary conditions in a bounded d-dimensional domain with smooth boundary. We apply the semigroup theory to obtain existence, uniqueness and maximal L2 regularity.

Jan Blechta, Preconditioning of H^1-regularized parameter identification. We consider the identification of spatially distributed parameters under H^1 regularization. Solving the associated minimization problem by Gauss–Newton iteration results in linearized problems to be solved in each step that can be cast as boundary value problems involving a low-rank modification of the Laplacian. Using algebraic multigrid as a fast Laplace solver, the Sherman–Morrison–Woodbury formula can be employed to construct a preconditioner for these linear problems which exhibits excellent scaling w.r.t. the relevant problem parameters. We first develop this approach in the functional setting, thus obtaining a consistent methodology for selecting boundary conditions that arise from the H^1 regularization. We then construct a method for solving the discrete linear systems based on combining any fast Poisson solver with the Woodbury formula. The efficacy of this method is then demonstrated with scaling experiments. These are carried out for a common nonlinear parameter identification problem arising in electrical resistivity tomography.

Andrea Cianchi, Optimal embeddings for symmetric gradient Orlicz-Sobolev spaces. A unified approach to embedding theorems for Sobolev-type spaces of vector-valued functions, defined via their symmetric gradient, is proposed. The Sobolev spaces in question are built upon general rearrangement-invariant norms. The case of symmetric gradient Orlicz-Sobolev spaces, of use in mathematical models in continuum mechanics driven by nonlinearities of non-power type, is especially focused. Optimal target spaces for embeddings of these spaces into Orlicz spaces (subcritical regime) or spaces of uniformly continuous functions (supercritical regime) are determined.

Lars Diening, Kac(an)ov iteration. We propose and investigate a iterative, linear method to solve the p-Laplace problem for large exponents. This extends earlier results for small exponents. The method should be applicable also to Non-Newtonian fluids.

Nicola Fusco, Regularity of capillarity droplets with obstacle. I will present a regularity result for Λ-minimizers of the capillarity energy in a half space with the wet part constrained to be confined in a given flat region. I will also provide some applications to a model of nanowire growth. The talk is based on a recent paper written in collaboration with Guido De Philippis and Massimiliano Morini.

Piotr Gwiazda, On a range of exponents for absence of Lavrentiev phenomenon for double phase functionals. For a class of functionals having the (p, q)-growth, we establish an improved range of exponents p, q for which the Lavrentiev phenomenon does not occur. The proof is based on a standard mollification argument and Young convolution inequality. Our contribution is two-fold. First, we observe that it is sufficient to regularise only bounded functions. Second, we exploit the L_∞ bound on the function rather than the Lp estimate on the gradient. Our proof does not rely on the properties of minimizers to variational problems but it is rather a consequence of the underlying Musielak-Orlicz function spaces. Moreover, our method works for unbounded boundary data, the variable exponent functionals and vectorial problems. In addition, the result seems to be optimal for $p \leq d$.

Jaroslav Hron, Computational study of hemodynamical flows in patient specific geometries. In this contribution, we give an overview of three particular cases of computational modelling of hemodynamical flows in patient specific geometries. We discuss the main modelling tasks from geometry segmentation to efficient numerical solution of the incompressible Navier-Stokes equations equipped with relevant boundary conditions that needs to come together to produce results that might be potentially helpful to some medical decision process.

Petr Kaplický, Uniqueness and regularity of flows of non-Newtonian fluids. We deal with the flows of non-Newtonian fluids in three dimensional setting subjected to the homogeneous Dirichlet boundary condition. Under the natural monotonicity, coercivity and growth condition on the Cauchy stress tensor expressed by a critical power index we show that weak solutions actually belong to a uniqueness class provided data of the problem are sufficiently smooth.

Ondřej Kreml, Wild solutions to isentropic Euler equations starting from smooth initial data. We consider the isentropic Euler equations of gas dynamics in the whole twodimensional space and we prove the existence of smooth initial data which admit infinitely many bounded admissible weak solutions. Taking advantage of the relation between smooth solutions to the Euler system and to the Burgers equation we construct a smooth compression wave which collapses into a perturbed Riemann state at some time instant $T > 0$. In order to continue the solution after the formation of the discontinuity, we apply the theory developed by De Lellis and Székelyhidi.

Martin Kružík, Nonlinear and linearized models in thermoviscoelasticity. We consider a quasistatic nonlinear model in thermoviscoelasticity at a finite-strain setting in the Kelvin–Voigt rheology where both the elastic and viscous stress tensors comply with the principle of frame indifference under rotations. The force balance is formulated in the reference configuration by resorting to the concept of nonsimple materials whereas the heat transfer equation is governed by the Fourier law in the deformed configurations. Weak solutions are obtained by means of a staggered in-time discretization where the deformation and the temperature are updated alternatingly. Afterward, we focus on the case of deformations near the identity and small temperatures, and we show by a rigorous linearization procedure that weak solutions of the nonlinear system converge in a suitable sense to solutions of a system in linearized thermoviscoelasticity. The same property holds for time-discrete approximations and we provide a corresponding commutativity result. This is joint work with R. Badal and M. Friedrich (Erlangen).

Vojtěch Kulvait, Coupled algorithms to solve propagation based phase contrast tomography. Computed tomography using coherent waves produced by a synchrotron must take into account the wave nature of X-rays. Diffraction caused by phase shift within the scanned material produces artefacts around sharp edges if conventional tomographic reconstruction methods are used. Therefore, it is necessary to compensate for the effects caused by wave propagation between the material and the detector. However, the combination of tomographic reconstruction and the wave backpropagation leads to a complicated nonlinear problem. In this contribution, I summarize different approaches to solve propagation based phase contrast tomography.

Martin Lanzendörfer, On shear-thinning fluids in porous media in estimating the pore size distribution. One interesting practical application of non-newtonain fluids in porous media is the possibility to estimate the pore size distribution of the media based on the porous media flow experiments with shear-thinning fluids. We will briefly review the recent results on two established methods and comment on different (practical and experimental, numerical) aspects of the problem.

Tomáš Los,On unsteady flows of viscoelastic fluids of the Giesekus type. Viscoelastic rate-type fluid models involving the stress and its observer-invariant time derivatives of higher order are used to describe the behaviour of materials with complex microstructure: geomaterials like asphalt, biomaterials such as vitreous in the eye, synthetic rubbers such as styrene butadiene rubber. A standard model that belongs to the category of viscoelastic rate-type fluid models of the second order is the model due to Burgers, which can be viewed as a mixture of two Oldroyd-B models of the first order. This viewpoint allows one to develop the whole hierarchy of generalized models of the Burgers type. We study one such generalization that can be viewed as a combination (mixture) of two Giesekus viscoelastic models having in general two different relaxation mechanisms. We prove long-time and large-data existence of weak solutions to the considered generalization of the Burgers model subject to no-slip boundary condition.

Daniel Lear, Traveling waves close to the Couette flow. In this talk we shall study the existence of smooth traveling waves close to the Couette flow for the 2D incompressible Euler equation for an ideal fluid. It is well known that this kind of solution does not exist arbitrarily close to the Couette flow if the distance is measured in Hs with $s > 3/2$, at the level of the vorticity. In this presentation we will deal with the case $s < 3/2$.

Václav Mácha, Local-in-time existence of strong solutions to a class of the compressible non-Newtonian Navier–Stokes equations. We show a local-in-time existence of a strong solution to the generalized compressible Navier-Stokes equations for arbitrarily large initial data. The goal is reached by L^p-theory for linearized equations which are obtained with help of the Weis multiplier theorem.

Josef Málek, On flows of viscoelastic rate type fluids with temperature dependent material coefficients. We derive a class of thermodynamically consistent variants of Oldroyd-B type models for incompressible viscoelastic fluids. In particular, we study the models that allow one to consider temperature dependent material coefficients. Apart from the incompressibility, homogeneity and isotropicity of the fluid, we assume a linear relation between the temperature and the internal energy, which is used is several applications. Yet, the temperature evolution equation contains terms that are ignored or even not thought of in most of the practically oriented (computational) works dealing with this class of fluids. We introduce a proper concept of weak solutions and establish their long-time and large-data existence. This is a joint work with Michal Bathory and Miroslav Bulíček.

Erika Maringová, On implicit constitutive relations. This is a joint work with Josef Málek and Miroslav Bulíček (Charles University). We study Navier-Stokes-like problem, where the stress tensor is related to the symmetric velocity gradient through an implicit equation. Formulating four conditions concerning the form of the implicit equation, we first show that these conditions describe a maximal monotone p-coercive graph. We then establish the global-in-time and large-data existence of a (weak) solution and its uniqueness. The theory is tractable from the point of view of numerical approximations.

Michal Pavelka, Multiscale and geometric mechanics and thermodynamics. Geometric mechanics (variational principles and Hamiltonian mechanics) expresses the evolution of classical particles and bodies. But Hamiltonian mechanics also describes the evolution of continuous systems (fluids, complex fluids, or solids). The Hamiltonian structure of continuum mechanics is advantageous because it starts from the first principles (quantum commutators or canonical particle mechanics) and then derives all the less detailed Hamiltonian structures on the various continuum levels. Eventually, the Hamiltonian structure also allows for structure-preserving numerical integrators like smoothed particle hydrodynamics for complex fluids.

Milan Pokorný, Homogenization of Navier–Stokes–Fourier system in domains with tiny holes. We consider the compressible Navier–Stokes–Fourier system in a domain with large number of holes. Under the assumption that the holes are sufficiently small, together with certain standard assumptions on the adiabatic exponent and the behaviour of the heat conductivity, we show that if passing simultaneously with the number of holes to infinity and their size to zero, in the limit we obtain again solution to the compressible Navier–Stokes–Fourier system in the domain without holes. The result holds both for the steady and evolutionary problem. The talk is based on a paper with Yong Lu (Nanjing University) and a paper with Emil Skříšovský (Charles University, Prague).

Dalibor Pražák, On linearization principle for the generalized NSE with dynamic boundary condition. We consider system of $d = 2$ or 3 equations in a bounded domain, coupled with a dynamic boundary condition of the type $\partial_t u + s(u) + [(Du)n]_t au = 0$, where n is the outer normal, τ denotes tangential projection and Du is symmetric velocity gradient. – We discuss how several classical results (linearized stability, estimates of the attractor dimension) generalize to our setting. (Joint work with B. Priyasad and M. Zelina)

Buddhika Priyasad, Finite-dimensional uniform stabilization of the Boussinesq system in Besov spaces. We consider the d-dimensional Boussinesq system defined on a sufficiently smooth bounded domain, with homogeneous boundary conditions, and subject to external sources, assumed to cause instability. The initial conditions for both fluid and heat equations are taken of low regularity. We then seek to uniformly stabilize such Boussinesq system in the vicinity of an unstable equilibrium pair, in the critical setting of correspondingly low regularity spaces, by means of explicitly constructed, feedback controls, which are localized on an arbitrarily small interior subdomain.

Vít Průša, A thermodynamic framework for non-isothermal phenomenological models of Mullins effect. The Mullins effect is a common name for a family of intriguing inelastic responses of various solid materials, in particular filled rubbers. Given the importance of the Mullins effect, there have been many attempts to develop mathematical models describing the effect. However, most of available models focus exclusively on the mechanical response, and are restricted to the idealised isothermal setting. We lift the restriction to isothermal processes, and we propose a full thermodynamic framework for a class of phenomenological models of the Mullins effect. In particular, we identify energy storage mechanisms (Helmholtz free energy) and entropy production mechanisms that on the level of stress–strain relation lead to the idealised Mullins effect or to the Mullins effect with permanent strain. The models constructed within the proposed framework can be used in the modelling of fully coupled thermo-mechanical processes, and the models are guaranteed to be consistent with the laws of thermodynamics.

Casey Rodriguez, Strain-limiting Cosserat rods with limiting small strain. In this talk, we introduce strain-limiting constitutive relations with limiting small strains for special Cosserat rods. We then discuss the approximate "small-strain" model wherein there is a nonlinear constitutive relation between the contact couple, contact force and small, linearized geometric strains. Finally, we discuss how solutions to the "small-strain" field equations faithfully approximate solutions to the fully nonlinear field equations. This talk is based on joint work with K. R. Rajagopal.

Ondřej Souček, Vapor plume on Enceladus – a modelling journey there and back again. In this talk, we present our recent understanding of the activity of the vapor plume emanating from the south-polar region of Saturn's moon Enceladus. The discovery of the plume lead to a skyrocketing increase of scientific interest in the moon as it brought evidence for the existence of its internal ocean and allowed for sampling of its composition. However, no satisfactory explanation has been provided so far concerning the observed timing of the plume activity. We present the results of our modeling effort in this regard combining a 3d visco-elasto-plastic model of the outer shell of the moon which includes Tiger stripes – cracks in the south polar region, a simplified hydrodynamic model of the water column dynamics in the cracks and a model of water vapor transport in the upper section of the crack. We discuss the capacity of the model to explain the mystery of Enceladus's plume activity timing.

Zdeněk Strakoš, A priori estimates of the spectrum and operator preconditioning. Many phenomena are mathematically expressed in terms of eigenvalues and eigenvectors of matrices and operators. Besides standard physical and engineering examples of waves and vibrations, they are essential also, e.g., in the mathematical foundations of quantum mechanics, which pioneered its use in spectral representations of Hermitian/self-adjoint operators. Consider a real symmetric n by n matrix G. It can be considered as a mapping that takes a vector u in the n-dimensional Euclidean space and maps it to a vector Gu in the same space. Basic linear algebra results state that there are exactly n vectors in this space that remain essentially unchanged when mapped by G, except for multiplication by a real number, i.e., $Gu_i = \lambda_i u_i$, $i = 1, 2, \ldots, n$. Moreover, these vectors are orthogonal, their normalized versions form an orthonormal basis and the matrix can be written as $G = \sum_{i=1}^{n} \lambda_i u_i u_i^T$, which is called the spectral decomposition of G. This result can be generalized to operators defined on an infinite dimensional real Hilbert space V. Indeed, any self-adjoint operator $\mathcal{G}\colon V \to V$ can be expressed in terms of the Riemann–Stieltjes integral as

$$\mathcal{G} = \int \lambda,\, \mathrm{d}E(\lambda), \quad \text{i.e.,} \quad (\mathcal{G}\psi, \phi) = \int \lambda,\, \mathrm{d}(E(\lambda)\psi, \phi); \forall \psi, \phi \in V,$$

for symmetric matrices. When such an infinite dimensional operator is discretized, as, e.g., when solving boundary value problems for partial differential equations, we should be concerned with the interplay between the eigenvalues of the matrices arising from discretizations and *the whole spectrum* of the associated infinite-dimensional operator. This issue is not only of theoretical interest, but it is also important for efficient numerical computations. Such consideration must naturally include the continuous part of the spectrum of $\mathcal{G}$. This contribution presents some recent results in this direction.

Agnieszka Świerczewska–Gwiazda, The Rayleigh–Benard problem for compressible fluid flows. We consider the physically relevant fully compressible setting of the Rayleigh–Benard problem of a fluid confined between two parallel plates, heated from the bottom, and subjected to the gravitational force. Under suitable restrictions imposed on the constitutive relations we show that this open system is dissipative in the sense of Levinson, meaning there exists a bounded absorbing set for any global–in–time weak solution. In the second part of the talk we discuss also the motion of a compressible viscous fluid in a container with impermeable boundary subject to time periodic heating and under the action of a time periodic potential force. We show the existence of a time periodic weak solution for arbitrarily large physically admissible data.

Tabea Tscherpel, Sobolev stability of the L^2-projection mapping to finite element spaces. Adaptively refined meshes are useful to approximate solutions of low regularity or to precisely track structures as the plug zone of Bingham or Herschel–Bulkley fluids. For such meshes it is challenging to achieve Sobolev stability of the L^2-projection mapping to Lagrange finite element spaces and it is indeed coupled to grading properties of the mesh. In the numerical analysis of time-dependent nonlinear problems the L^2 projection and $W^{1,p}$-stability (for general p) thereof is a crucial tool. We present stability results for 2D and 3D which apply to a large class of adaptive refinement schemes.

Karel Tůma, Viscoelastic rate-type fluids: thermodynamically compatible derivation, applications and simulations. In this talk, we deal with different aspects of viscoelastic rate-type fluids. First, we show a thermodynamic derivation of a few different rate-type fluid models including the classical models such as Maxwell, Oldroyd-B, or Burgers satisfying automatically the second law of thermodynamics. Next, we show that they can be used to capture the experimental data for various geomaterials such as asphalt binder/concrete or biomaterials such as the vitreous. The appropriate models are then implemented using finite element methods and with fitted material parameters the corresponding boundary-value problems are solved. Finally, on the example of the rebound of an elastic body in a viscous incompressible fluid, we show how the viscoelastic model can be used to solve fluid-structure interaction problems in a fully Eulerian setting.

Alexis Vasseur, Boundary vorticity estimate for the Navier-Stokes equation and control of layer separation in the inviscid limit. We provide a new boundary estimate on the vorticity for the incompressible Navier-Stokes equation endowed with no-slip boundary conditions. The estimate is scalable through the inviscid limit. It controls the layer separation at the inviscid Kato double limit, consistent with the Layer separation predictions via convex integration.

Jakub Woźnicki, Non-Newtonian fluids with discontinuous-in-time stress tensor. We consider the system of equations describing the flow of incompressible fluids in bounded domain. In the considered setting, the Cauchy stress tensor is a monotone mapping and has asymptotically $(s-1)$-growth with the parameter s depending on the spatial and time variable. We do not assume any smoothness of s with respect to time variable and assume the log-Holder continuity with respect to spatial variable. Such a setting is a natural choice if the material properties are instantaneous, e.g. changed by the switched electric field. We establish the long time and the large data existence of weak solution provided that $s \geq (3d+2)/(d+2)$.

Michael Zelina, On the uniqueness of the solution and finite-dimensional attractors for the 3D flow with dynamic slip boundary condition. This is a joint work with Dalibor Pražák. We consider a general class of incompressible, non-Newtonian fluids of power-law model occupying a bounded domain in $\mathbb{R}^3$. We work with a rather general class of the so-called dynamic boundary condition of the type $\partial_t u + s(u) = -[Sn]_\tau$, where the quantity s is possibly implicitly related to u. Our aim here is to extend some results which are well-known for the same model subject to a homogenenous Dirichlet boundary condition. Firstly, we use the recent approach of Nikolskii spaces to obtain additional time regularity of arbitrary weak solution, provided that the power-law exponent is at least equal to the critical value 11/5. As a second and natural corollary, we establish the existence of global attractor for all these exponents. Assuming further that the boundary nonlinearity s is represented as a function of polynomial growth in u, we show that the attractor is finite-dimensional.