Sharp Interface Limit of a Navier-Stokes/Allen-Cahn System with Vanishing Mobility

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Overview

1. Sharp Interface Limit for a Navier-Stokes/Allen-Cahn System

2. Relative Entropy Method for $m_\varepsilon = m_0\varepsilon^k$, $k \in (0,2)$

Overview

Diffuse Interface Models for Two-Phase Flows of Incompressible Fluids

We assume $\rho \equiv const.$ and consider

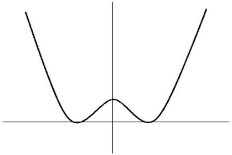
$$\rho(\partial_t \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) - \operatorname{div}(2\nu(c_\varepsilon) D\boldsymbol{v}_\varepsilon) + \nabla p_\varepsilon = -\varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \tag{NSt1}$$

$$\operatorname{div} \boldsymbol{v}_\varepsilon = 0 \tag{NSt2}$$

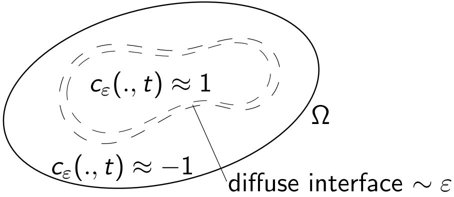
coupled with an Allen-Cahn equation

$$\partial_t c_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla c_\varepsilon = -\frac{m_\varepsilon}{\varepsilon} \underbrace{\left(-\varepsilon \Delta c_\varepsilon + \tfrac{1}{\varepsilon} f'(c_\varepsilon)\right)}_{= DE_\varepsilon(c_\varepsilon) =: \mu_\varepsilon}, \tag{AC}$$

cf. e.g. Jiang, Li & Liu '17 together with suitable boundary and initial conditions, where $f : \mathbb{R} \to \mathbb{R}$ is a suitable double well potential and $m_\varepsilon > 0$.



Example: $f(c) = \frac{1}{8}(1 - c^2)^2$



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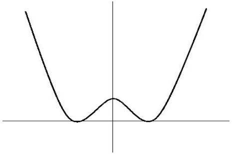

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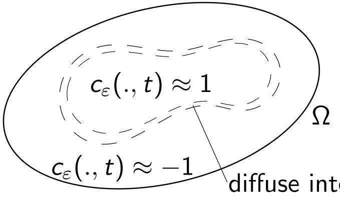

Question: Convergence of these models as $\varepsilon \to 0$ in dependence on $m_\varepsilon = m_0 \varepsilon^k$, $m_0 > 0$, $k \geq 0$?

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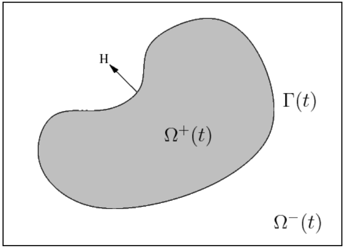
Energy dissipation: For smooth solutions one has

$$\begin{aligned}
\frac{d}{dt} E(c_\varepsilon(t), \mathbf{v}_\varepsilon(t)) &= -\int_\Omega \nu(c_\varepsilon) |D\mathbf{v}_\varepsilon|^2 \, dx - \int_\Omega \frac{m_\varepsilon}{\varepsilon} |\mu_\varepsilon|^2 \, dx \quad \text{with} \\
E(c_\varepsilon(t), \mathbf{v}_\varepsilon(t)) &= E_\varepsilon(c_\varepsilon(t)) + \int_\Omega \rho \frac{|\mathbf{v}_\varepsilon(x,t)|^2}{2} \, dx, \\
E_\varepsilon(c_\varepsilon(t)) &= \frac{\varepsilon}{2} \int_\Omega |\nabla c_\varepsilon(x,t)|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega f(c_\varepsilon(x,t)) \, dx
\end{aligned}$$

Formal Asymptotics for Navier-Stokes/Allen-Cahn system

Bulk equations: In $\Omega^\pm(t)$ we have

$$\rho\partial_t \boldsymbol{v} + \rho \boldsymbol{v}\cdot\nabla \boldsymbol{v} - \operatorname{div}(2\nu^\pm D\boldsymbol{v}) + \nabla p = 0$$
$$\operatorname{div} \boldsymbol{v} = 0$$



Interface equations: Case I: $m_\varepsilon = \varepsilon m_0$: On Γ_t we have

$$-\left[\boldsymbol{n}_{\Gamma_t}\cdot(2\nu^\pm D\boldsymbol{v} - p\mathbf{I})\right] = \sigma H \boldsymbol{n}_{\Gamma_t}$$
$$V_{\Gamma_t} = \boldsymbol{n}_{\Gamma_t}\cdot \boldsymbol{v}|_{\Gamma_t}$$

where $[u](x) = \lim_{\varepsilon\to 0+}(u(x+\varepsilon\boldsymbol{n}_{\Gamma_t}) - u(x-\varepsilon\boldsymbol{n}_{\Gamma_t}))$.

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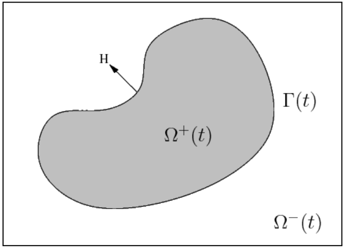

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where $[u](x) = \lim_{\varepsilon\to 0+}(u(x+\varepsilon\boldsymbol{n}_{\Gamma_t}) - u(x-\varepsilon\boldsymbol{n}_{\Gamma_t}))$.

Case II: $m_\varepsilon = m_0 > 0$: On Γ_t we have

$$\begin{aligned} -\left[\boldsymbol{n}_{\Gamma_t}\cdot(2\nu^\pm D\boldsymbol{v} - p\mathbf{I})\right] &= \sigma H_{\Gamma_t}\boldsymbol{n}_{\Gamma_t} \\ V_{\Gamma_t} &= \boldsymbol{n}_{\Gamma_t}\cdot\boldsymbol{v}|_{\Gamma_t} + m_0 H_{\Gamma_t}, \end{aligned}$$

cf. A. '22 together with A., Garcke, Grün '12. Here $\sigma = \int_{-1}^{1}\sqrt{2f(s)}\,ds$.

Overview of Rigorous Analytic Results ($m_\varepsilon = m_0\varepsilon^k$)

Asymptotic Expansion Method (De Mottoni, Schatzman '89 for Allen-Cahn equation):

- A. & Y. Liu '18: Convergence for small times with convergence rates in $\mathbb{T}^2$ for Stokes/Allen-Cahn system with same viscosities and $k = 0$.
- A. & Fei '22: Extension to Navier-Stokes/Allen-Cahn system with variable viscosities and $k = 0$, $d = 2$.
- A., Fei, Moser '23: Convergence for Navier-Stokes/Allen-Cahn system with variable viscosities and $k = \frac{1}{2}$, $d = 2$.

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Relative Entropy Method (Fischer, Laux, Simon '20 for Allen-Cahn eq.):

- S. Hensel & Y. Liu '22: Convergence in $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, for Navier-Stokes/Allen-Cahn system with same viscosities and $k = 0$ using a relative entropy method.
- A., Fischer, Moser '23: Convergence for Navier-Stokes/Allen-Cahn system with constant viscosities and $k \in (0, 2)$, $d = 2, 3$ using a relative entropy method.

Remark: There is a counterexample for convergence if $m_\varepsilon = o(\varepsilon^2)$ with inflow boundary condition. (A. '22 together with A. & Lengeler '14)

Preliminaries: Signed Distance Function

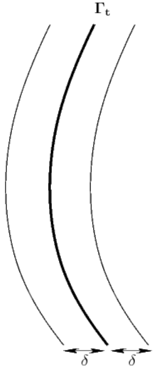
Moreover, let

$$d_{\Gamma_t}(x) = \begin{cases} \operatorname{dist}(x, \Gamma_t) & \text{if } x \in \overline{\Omega^+(t)} \\ -\operatorname{dist}(x, \Gamma_t) & \text{if } x \in \Omega \setminus \overline{\Omega^+(t)} =: \Omega^-(t) \end{cases}$$

be the signed distance function to Γ_t.

Remark: If Γ_t is at least C^2, there is some $\delta > 0$ such that d_{Γ_t} is as smooth as Γ_t on

$$\Gamma_t(\delta) := \{x \in \mathbb{R}^d : \operatorname{dist}(x, \Gamma_t) < \delta\}.$$



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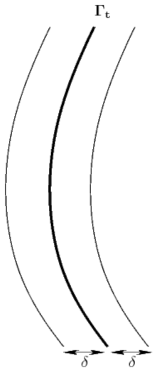

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Important properties:

$$\partial_t d_{\Gamma_t} = -V_{\Gamma_t}, \quad \nabla d_{\Gamma_t} = \boldsymbol{n}_{\Gamma_t}, \quad \Delta d_{\Gamma_t} = -H_{\Gamma_t} \quad \text{on } \Gamma_t,$$

where V_{Γ_t} is the normal velocity, H_{Γ_t} is the mean curvature, $\boldsymbol{n}_{\Gamma_t}$ is a normal. Finally, let $P_{\Gamma_t} : \Gamma_t(\delta) \to \Gamma_t$ be the orthogonal projection onto Γ_t.

Overview

Sharp Interface Limit for a Navier-Stokes/Allen-Cahn System

Let $m_\varepsilon = m_0\varepsilon^k$, $k \in (0,2)$. We consider the sharp interface limit $\varepsilon \to 0$ for

$$\partial_t \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon - \operatorname{div}(2\nu(c_\varepsilon) D\boldsymbol{v}_\varepsilon) + \nabla p_\varepsilon = -\varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon), \tag{NSAC1}$$

$$\operatorname{div} \boldsymbol{v}_\varepsilon = 0, \tag{NSAC2}$$

$$\partial_t c_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla c_\varepsilon = m_\varepsilon\left(\Delta c_\varepsilon - \tfrac{1}{\varepsilon^2} f'(c_\varepsilon)\right), \tag{NSAC3}$$

in $\Omega \times [0, T_0]$, which formally converges to

$$\begin{aligned}
\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nu^\pm \Delta \boldsymbol{v} + \nabla p &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0],\\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0],\\
-\left[\boldsymbol{n}_{\Gamma(t)} \cdot (2\nu^\pm D\boldsymbol{v} - pI)\right] &= \sigma H_{\Gamma(t)} \boldsymbol{n}_{\Gamma(t)} && \text{on } \Gamma(t), t \in [0, T_0],\\
V_{\Gamma(t)} - \boldsymbol{n}_{\Gamma(t)} \cdot \boldsymbol{v}|_{\Gamma(t)} &= 0 && \text{on } \Gamma(t), t \in [0, T_0]
\end{aligned}$$

in a bounded, smooth domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, together with Dirichlet boundary conditions.

Relative Entropy Method

Idea: construct energy/entropy-like functionals with suitable coercivity properties and error control. Show Gronwall-type estimate

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Idea: construct energy/entropy-like functionals with suitable coercivity properties and error control. Show Gronwall-type estimate

Idea goes back to conservation laws (Dafermos '79, DiPerna '79), recently adapted for curvature driven interface evolution problems by Fischer, Hensel, Laux, Simon. Some results:

- Fischer, Hensel '20: Weak-strong uniqueness for two-phase Navier-Stokes equation with surface tension
- Fischer, Laux, Simon '20: Sharp interface limit for Allen-Cahn equation towards mean curvature flow
- Hensel, Liu '22: Navier-Stokes/Allen Cahn with constant mobility ($k = 0$ above) towards Navier-Stokes/mean curvature flow system

Theorem (A., Fischer, Moser '23)

Let $m_\varepsilon := m_0\varepsilon^k > 0$, *where* $m_0 > 0$ *and* $k \in (0, 2)$ *are fixed,* $d = 2, 3$ *and*

1. *Let* $T_0 > 0$ *be such that the two-phase Navier-Stokes system with surface tension has a smooth solution* $(\boldsymbol{v}, p, \Gamma)$ *on* $[0, T_0]$.
2. *Let* $(\boldsymbol{v}_\varepsilon, p_\varepsilon, c_\varepsilon)$ *be weak solutions to Navier-Stokes/Allen-Cahn on* $[0, T_0]$ *for* $\varepsilon > 0$ *small, mobility* m_ε *and for well-prepared initial data (i.e. entropy functionals small at initial time with certain rate).*

Then for $\varepsilon > 0$ *small and a.e.* $T \in [0, T_0]$ *it holds*

$$\|(\boldsymbol{v}_\varepsilon - \boldsymbol{v})(., T)\|_{L^2(\Omega)} + \|\sigma\chi_{\Omega_T^+} - \psi_\varepsilon(., T)\|_{L^1(\Omega)} \leq C\left(\frac{\varepsilon}{\sqrt{m_\varepsilon}} + m_\varepsilon\right),$$

where $\psi_\varepsilon := \psi \circ c_\varepsilon$ *with* $\psi(r) := \int_{-1}^{r} \sqrt{2f(s)}\, ds$.

Remark: In the case $k > 2$ there is a counter-example for convergence. (A. '22 together with A. & Lengeler '14)

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Idea: compare with strong solution $(\boldsymbol{v}^{m_\varepsilon}, p^{m_\varepsilon}, \Gamma^{m_\varepsilon})$ to a modified two-phase Navier-Stokes system, where interface evolution is replaced by

$$V_{\Gamma_t^{m_\varepsilon}} = \boldsymbol{n}_{\Gamma_t^{m_\varepsilon}} \cdot \boldsymbol{v}_{m_\varepsilon}^{\pm} + m_\varepsilon H_{\Gamma_t^{m_\varepsilon}} \quad \text{on } \Gamma_t^{m_\varepsilon}, t \in [0, T_0].$$

Relative Energy Functionals

- Notation: $\boldsymbol{n}$ (by projection extended) normal of Γ^{m_ε}, $d_{\Gamma^{m_\varepsilon}}$ signed distance of Γ^{m_ε} and $\Gamma^{m_\varepsilon}(\delta)$ tubular neighbourhood, $\delta > 0$ small.
- We define the relative entropy functional as

$$E[\boldsymbol{v}_\varepsilon, c_\varepsilon | \boldsymbol{v}^{m_\varepsilon}, \Gamma^{m_\varepsilon}](t) := \int_\Omega \frac{1}{2}|\boldsymbol{v}_\varepsilon - \boldsymbol{v}^{m_\varepsilon}|^2(., t)\, dx + E[c_\varepsilon|\Gamma^{m_\varepsilon}](t),$$

$$E[c_\varepsilon|\Gamma^{m_\varepsilon}](t) := \int_\Omega \frac{\varepsilon}{2}|\nabla c_\varepsilon|^2(., t) + \frac{1}{\varepsilon} f(c_\varepsilon(., t)) - (\xi \cdot \nabla \psi_\varepsilon)(., t)\, dx,$$

where $\xi := \overline{\eta}(\frac{d_{\Gamma^{m_\varepsilon}}}{\delta})\boldsymbol{n}$ and $\overline{\eta}$ is a cutoff with quadratic decay and $\psi_\varepsilon := \psi \circ c_\varepsilon$.

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- We define the bulk error functional by

$$E_{\text{bulk}}[c_\varepsilon|\Gamma^{m_\varepsilon}](t) := \int_\Omega \left(\sigma \chi_{\Omega_t^{m_\varepsilon,+}} - \psi_\varepsilon(.,t)\right) \vartheta(.,t)\,dx,$$

where $\vartheta : \overline{\Omega} \times [0, T_0] \to [0,1]$ is smooth, proportional to $d_{\Gamma^{m_\varepsilon}}$ close to Γ^{m_ε} and cut off to ± 1 outside.

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- Goal: Gronwall-type estimate for $E[\boldsymbol{v}_\varepsilon, c_\varepsilon | \boldsymbol{v}^{m_\varepsilon}, \Gamma^{m_\varepsilon}] + E_{\text{bulk}}[c_\varepsilon|\Gamma^{m_\varepsilon}]$. Show and use coercivity properties.

Lemma (cf. Fischer, Laux, Simon '20)

For every $t \in [0, T]$ we have

$$\int_\Omega |\boldsymbol{n}_\varepsilon - \boldsymbol{\xi}|^2 (|\nabla \psi_\varepsilon| + \varepsilon |\nabla c_\varepsilon|^2)\, dx \leq CE[c_\varepsilon|\Gamma^{m_\varepsilon}] \text{ (tilt-excess type error)} \tag{1}$$

$$\int_\Omega \left(\sqrt{\varepsilon} |\nabla c_\varepsilon| - \tfrac{1}{\sqrt{\varepsilon}} \sqrt{2f(c_\varepsilon)} \right)^2 dx \leq 2E[c_\varepsilon|\Gamma^{m_\varepsilon}] \text{ (error in equipartition)} \tag{2}$$

$$\int_\Omega \min\{d_{\Gamma_t}^2, 1\} \left(\varepsilon |\nabla c_\varepsilon|^2 + |\nabla \psi_\varepsilon| \right) dx \leq CE[c_\varepsilon|\Gamma^{m_\varepsilon}] \text{ (error far from interface)} \tag{3}$$

for some $C > 0$, where $\boldsymbol{n}_\varepsilon = \frac{\nabla c_\varepsilon}{|\nabla c_\varepsilon|}$ if $\nabla c_\varepsilon \neq 0$.

Lemma (cf. Fischer, Laux, Simon '20)

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Proof.

The starting point is

$$E[c_\varepsilon|\Gamma^{m_\varepsilon}] = \int_\Omega \frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} f(c_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon\, dx = \int_\Omega \frac{1}{2}\left(\sqrt{\varepsilon}|\nabla c_\varepsilon| - \tfrac{1}{\sqrt{\varepsilon}}\sqrt{2f(c_\varepsilon)}\right)^2 dx + \int_\Omega (1 - \boldsymbol{n}_\varepsilon \cdot \boldsymbol{\xi})|\nabla \psi_\varepsilon|\, dx,$$

where $2(1 - \boldsymbol{n}_\varepsilon \cdot \boldsymbol{\xi}) \geq |\boldsymbol{n}_\varepsilon - \boldsymbol{\xi}|^2$ since $|\xi| \leq 1$. This implies (2) and half of (1). □

Gronwall-Type Estimates

With $H_\varepsilon := -\varepsilon\Delta c_\varepsilon + \frac{1}{\varepsilon}f'(c_\varepsilon)$ we have:

$$
\begin{aligned}
E[\mathbf{v}_\varepsilon, c_\varepsilon|\mathbf{v}^{m_\varepsilon}, \Gamma^{m_\varepsilon}](T) \leq\ & E[\mathbf{v}_\varepsilon, c_\varepsilon|\mathbf{v}^{m_\varepsilon}, \Gamma^{m_\varepsilon}](0) - \int_0^T \int_\Omega |\nabla \mathbf{v}_\varepsilon - \nabla \mathbf{v}^{m_\varepsilon}|^2\,dx\,dt \\
& - \int_0^T \int_\Omega \frac{m_\varepsilon}{2\varepsilon} \left| H_\varepsilon + \sqrt{2f(c_\varepsilon)}\nabla\cdot\xi \right|^2 dx\,dt - \int_0^T \int_\Omega \frac{m_\varepsilon}{2\varepsilon} \left| H_\varepsilon - \frac{\mathbf{B}-\mathbf{v}^{m_\varepsilon}}{m_\varepsilon}\cdot\xi\,\varepsilon|\nabla c_\varepsilon| \right|^2 dx\,dt \\
& - \int_0^T \int_\Omega (\mathbf{v}_\varepsilon - \mathbf{v}^{m_\varepsilon})\cdot((\mathbf{v}_\varepsilon - \mathbf{v}^{m_\varepsilon})\cdot\nabla)\mathbf{v}^{m_\varepsilon}\,dx\,dt \\
& - \int_0^T \int_\Omega ((\partial_t + \mathbf{B}\cdot\nabla)|\xi|^2)\,|\nabla\psi_\varepsilon|\,dx\,dt \\
& + \int_0^T \int_\Omega m_\varepsilon \left| \frac{\mathbf{B}-\mathbf{v}^{m_\varepsilon}}{m_\varepsilon}\cdot\xi + \nabla\cdot\xi \right|^2 \varepsilon|\nabla c_\varepsilon|^2\,dx\,dt \\
& - \int_0^T \int_\Omega \frac{1}{\sqrt{\varepsilon}} \left(H_\varepsilon + \sqrt{2f(c_\varepsilon)}\nabla\cdot\xi \right) (\mathbf{v}^{m_\varepsilon} - \mathbf{B})\cdot(\mathbf{n}_\varepsilon - \xi)\sqrt{\varepsilon}|\nabla c_\varepsilon|\,dx\,dt \\
& - \int_0^T \int_\Omega \xi\otimes\xi : \nabla\mathbf{B}(\varepsilon|\nabla c_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\,dx\,dt + \ldots
\end{aligned}
$$

Choice of Field $\mathbf{B}$

$\mathbf{B}$ should approximately transport and rotate ξ in the sense that

$$|(\partial_t + \mathbf{B}\cdot\nabla)|\xi|^2| \le C\min\{d^2_{\Gamma^{m_\varepsilon}}, 1\} \quad \text{a.e. in } \Omega\times[0,T],$$
$$|\partial_t\xi + (\mathbf{B}\cdot\nabla)\xi + (\nabla B)^\top\xi| \le C\min\{d_{\Gamma^{m_\varepsilon}}, 1\} \quad \text{a.e. in } \Omega\times[0,T].$$

and such that the following three problematic terms are controlled:

1. $\int_0^T\int_\Omega m_\varepsilon \left|\frac{\mathbf{B}-\boldsymbol{v}^{m_\varepsilon}}{m_\varepsilon}\cdot\xi + \nabla\cdot\xi\right|^2 \varepsilon|\nabla c_\varepsilon|^2\,dx\,dt$
2. $\int_0^T\int_\Omega \frac{1}{\sqrt{\varepsilon}}\left(H_\varepsilon + \sqrt{2f(c_\varepsilon)}\nabla\cdot\xi\right)(\boldsymbol{v}^{m_\varepsilon} - \mathbf{B})\cdot(\boldsymbol{n}_\varepsilon - \xi)\sqrt{\varepsilon}|\nabla c_\varepsilon|\,dx\,dt$
3. $\int_0^T\int_\Omega \xi\otimes\xi : \nabla\mathbf{B}(\varepsilon|\nabla c_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\,dx\,dt$

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We choose

$$\mathbf{B} := \boldsymbol{v}^{m_\varepsilon} + m_\varepsilon H\boldsymbol{n}\,\tilde{\eta}\left(\frac{d_{\Gamma^{m_\varepsilon}}}{\delta}\right),$$

where H is the (by projection extended) mean curvature of Γ^{m_ε} and $\tilde{\eta}$ is a plateau cutoff.
Then term 3 remains as last problematic term!

The last problematic term

How we estimate $\int_0^T \int_\Omega \xi \otimes \xi : \nabla \mathbf{B}(\varepsilon |\nabla c_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \, dx \, dt$?

Idea:

- Write $\xi \otimes \xi : \nabla \mathbf{B} = \partial_n \eta$ (normal derivative),

$$\eta(x,t) := \int_{h_\varepsilon(P_{\Gamma^{m_\varepsilon}}(x,t),t)}^{d_{\Gamma^{m_\varepsilon}}(x,t)} \xi \otimes \xi : \nabla \mathbf{B}|_{(P_{\Gamma^{m_\varepsilon}}(x,t)+r\boldsymbol{n}(P_{\Gamma^{m_\varepsilon}}(x,t),t),t)} \, dr,$$

where P^{m_ε} is projection onto Γ^{m_ε} and h_ε is some height function.

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- The height function $h_\varepsilon(.,t)$ is constructed to be Lipschitz and approximates the boundary of a suitable level set of $c_\varepsilon(.,t)$.
- With an ODE comparison argument one can control the energy $\varepsilon\frac{|\nabla c_\varepsilon|^2}{2} + \frac{f(c_\varepsilon)}{\varepsilon}$ away from a small strip around the $h_\varepsilon(.,t)$-graph.
- Together with integration by parts and coercivity properties of $E[c_\varepsilon|\Gamma^{m_\varepsilon}](t) + E_{\text{bulk}}[c_\varepsilon|\Gamma^{m_\varepsilon}](t)$ we can estimate the remaining term

Existence for the Approximate Two-Phase flow

Important: Existence of strong solutions $(\mathbf{v}_m^\pm, p_m^\pm, \Gamma^m)$ for the modified two-phase flow

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\begin{aligned}
\partial_t \mathbf{v}_m^\pm + \mathbf{v}_m^\pm \cdot \nabla \mathbf{v}_m^\pm - \Delta \mathbf{v}_m^\pm + \nabla p_m^\pm &= 0 && \text{in } \Omega_t^{m,\pm}, t \in [0, T_0], && (4)\\
\operatorname{div} \mathbf{v}_m^\pm &= 0 && \text{in } \Omega_t^{m,\pm}, t \in [0, T_0], && (5)\\
- [\![2D\mathbf{v}_m^\pm - p_m^\pm \mathbf{I}]\!] \mathbf{n}_{\Gamma_t^m} &= \sigma H_{\Gamma_t^m} \mathbf{n}_{\Gamma_t^m} && \text{on } \Gamma_t^m, t \in [0, T_0], && (6)\\
[\![\mathbf{v}_m^\pm]\!] &= 0 && \text{on } \Gamma_t^m, t \in [0, T_0], && (7)\\
V_{\Gamma_t^m} - \mathbf{n}_{\Gamma_t^m} \cdot \mathbf{v}_m^\pm &= m H_{\Gamma_t^m} && \text{on } \Gamma_t^m, t \in [0, T_0], && (8)\\
\mathbf{v}_m^-|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0, T_0), && (9)\\
\Gamma_0^m = \Gamma^0, \quad \mathbf{v}_m^\pm|_{t=0} &= \mathbf{v}_0^\pm && \text{in } \Omega_0^\pm, && (10)
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for sufficiently small $m > 0$.

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for sufficiently small $m > 0$.

Idea: Look for a solution of (4)-(10) such that $\Gamma_t^m = \Phi_{h_m}(\Gamma_t)$ for all $t \in [0, T_0]$ a sufficiently regular, small $h_m \colon \Gamma \to \mathbb{R}$, where $\Phi_.$ is a Hanzawa transformation with respect to $\Omega^\pm(t)$, $t \in [0, T_0]$.

Then $(\boldsymbol{v}_m^\pm, p_m^\pm, (\Gamma_t^m)_{t\in[0,T_0]})$ solves (4)-(10) if and only if

$$\boldsymbol{w}_m^\pm(x,t) := \boldsymbol{v}_m^\pm(\Theta_h(x,t),t), \quad q^\pm(x,t) = p_m^\pm(\Theta_h(x,t),t) \quad \text{for } x \in \Omega_t^\pm, t \in [0, T_0]$$

solves a transformed system

$$\begin{aligned}
\partial_t \boldsymbol{w}_m^\pm - \Delta \boldsymbol{w}_m^\pm + \nabla q_m^\pm &= \mathbf{f}_m^\pm(h_m, \boldsymbol{w}^\pm, q^\pm) && \text{in } \Omega^\pm, && (11)\\
\operatorname{div} \boldsymbol{w}_m^\pm &= g(h_m)\boldsymbol{v}^\pm && \text{in } \Omega^\pm, && (12)\\
[\![\boldsymbol{w}_m]\!] &= 0 && \text{on } \Gamma, && (13)\\
[\![2D\boldsymbol{w}_m^\pm - q^\pm \mathbf{I}]\!]\boldsymbol{n}_{\Gamma_t} &= \mathbf{a}(h_m, \boldsymbol{w}_m, q_m) && \text{on } \Gamma, && (14)\\
\boldsymbol{w}_m^-|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0, T_0), && (15)\\
\partial_t^\bullet h_m - \boldsymbol{n}_{\Gamma_t} \cdot \boldsymbol{w}_m &= m\Delta_{\Gamma_t} h_m + b(h_m, \boldsymbol{w}_m) && \text{on } \Gamma, && (16)\\
\boldsymbol{w}|_{t=0} &= \boldsymbol{v}_0 && \text{on } \Omega_0^\pm, && (17)
\end{aligned}$$

Theorem

Let $q > d + 2$. There is some $m_0 > 0$ such that for every $m \in (0, m_0]$ the transformed system (11)-(17) possesses a solution $(\boldsymbol{v}, p, [\![p]\!], h) \in \mathbb{E}_m(T_0)$, which satisfies

$$\|(\boldsymbol{v} - \boldsymbol{v}_0, p - p_0, [\![p - p_0]\!], h)\|_{\mathbb{E}_m(T_0)} \leq Cm$$

for some $C > 0$ independent of $m \in (0, m_0]$.

Here $\mathbb{E}_m(T_0) := \mathbb{E}_1(T_0) \times \mathbb{E}_2(T_0) \times \mathbb{E}_3(T_0) \times \mathbb{E}_{4,m}(T_0)$,

$$\mathbb{E}_1(T_0) := {}_0W^1_q(0, T_0; L^q(\Omega))^d \cap L^q(0, T_0; W^2_q(\Omega \setminus \Gamma_t))^d, \quad \mathbb{E}_2(T_0) = \ldots, \quad \mathbb{E}_3(T_0) = \ldots,$$

$$\mathbb{E}_{4,m}(T_0) := W_q^{2-\frac{1}{2q}}(0, T_0; L^q(\Gamma_t)) \cap {}_0W^1_q(0, T_0; W_q^{2-\frac{1}{q}}(\Gamma_t)) \cap L^q(0, T_0; W_q^{4-\frac{1}{q}}(\Gamma_t)),$$

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where the norm of $\mathbb{E}_{4,m}(T_0)$ depends on m suitably.

Steps of proof:

1. Show maximal L^q-regularity of linearized system uniformly in $m > 0$.
2. Apply contraction mapping principle.

Thank you for your attention!

Main References:

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