Department of Mechanical Engineering
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Dynamic Fragmentation of Functionally Graded Brittle Materials in 1D

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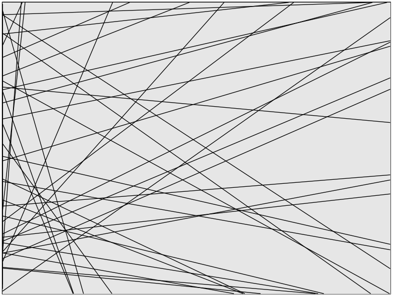
Outline

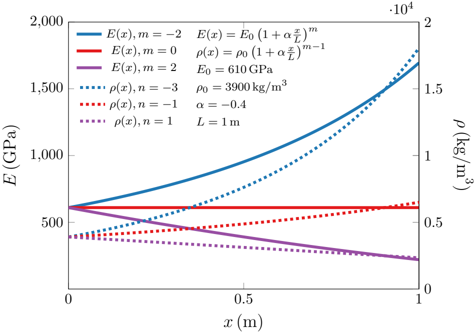
- **Fragmentation**
  A contiguous body breaks into several pieces when loaded beyond design capacity.
- **Application**
  Fragments are desired (e.g., kidney stones), prevented (e.g., bulletproof vests), or controlled (e.g., rock mining).
- **Analytical Modelling**
  Predicts minimum fragment size in materials under constant dynamic strain rates.



$\boxed{Q}$ **Can fragmentation in brittle materials be tailored?**

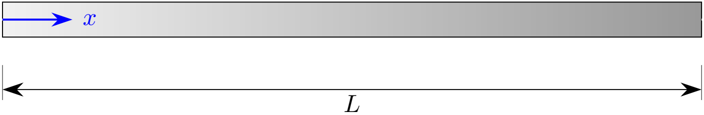
Functionally Graded Brittle Materials

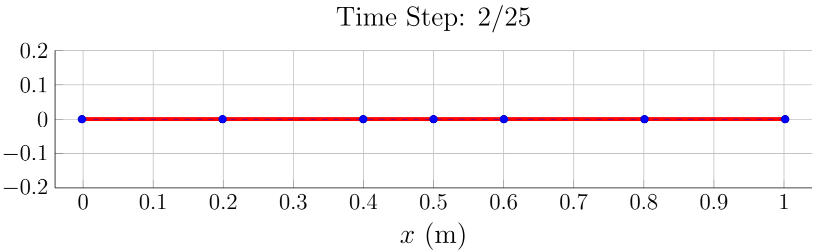
- **F**untionally **G**raded **M**aterials
  Spatial variation in the material properties.
- Engineering Applications
  High fidelity & robustness.
- Testing & Characterization
  Elastic wave propagation in FGM's
  - Reliable NDT methods
  - Assess structural integrity



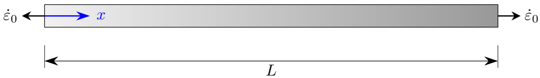
$\boxed{Q}$ **Can we design FGMs with tailored wave propagation characteristics to control fragmentation behaviour?**

Model Formulation





Model Formulation



- **Governing Equation**

$$\left(\hat{E}(x)u_{,x}\right)_{,x} = \rho(x)u_{,tt} \quad \text{where} \quad c(x) = \sqrt{\hat{E}(x)/\rho(x)}$$

- **Initial Conditions**

$$u(x,0) = 0, \quad u_{,t}(x,0) = \dot{\varepsilon}_0\left(x - \frac{L}{2}\right)$$

- **Displacement Field (Transcendental Form)**

For material properties exhibiting a polynomial variation:

$$\hat{U}(z,s) = C_1 I_0\left(\frac{2s}{\alpha}\sqrt{z}\right) + C_2 K_0\left(\frac{2s}{\alpha}\sqrt{z}\right) + \frac{\dot{\varepsilon}_0}{s^2\alpha}z + \frac{\dot{\varepsilon}_0}{s^4\alpha}(\alpha^2 - s^2)$$

where $z = \left(1 + \frac{\alpha x}{L}\right)$, $I_0\left(\frac{2s}{\alpha}\sqrt{z}\right)$, and $K_0\left(\frac{2s}{\alpha}\sqrt{z}\right)$ are the modified Bessel functions of the first and second kinds, respectively.

$\boxed{Q}$ **Can we get a closed-form displacement field $u(x,t)$?**

Homotopy Perturbation Method (HPM)

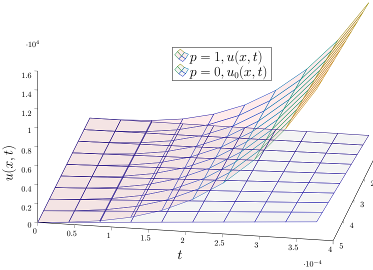
Let $u = u_0 + pu_1 + p^2u_2 + \dots$

- Governing Equation:

$$u_{,tt} = p\{c_0^2(1+\frac{\alpha x}{L})u_{,xx} + c_0^2\frac{\alpha x}{L}u_{,x}\}$$

where:

$$u = \lim_{p\to 1}(u_0 + pu_1 + p^2u_2 + \dots)$$



Form	$\hat{E}(x) = E_0\times$	$\rho(x) = \rho_0\times$	Cases	$u(x,t)$
Polynomial	ψ^m	ψ^n	$m-n=1$	$\dot{\varepsilon}_0 t\,(x - L/2 + m\phi)$
			$m-n=2$	$\frac{\dot{\varepsilon}_0\psi L^2}{\alpha^2\sqrt{m}c_0}\sinh\left(\frac{\sqrt{m}c_0\alpha t}{L}\right) - \frac{\dot{\varepsilon}_0 tL}{\alpha}$
Sinusoidal	$\sin^m(\beta(x)-\alpha)$	$\sin^n(\beta(x)-\alpha)$	$m-n=0$ $(m=n=2)$	$\dot{\varepsilon}_0 t\,(x - L/2 + m\phi\cot(\beta(x)-\alpha))$
Exponential	$\exp(m\beta(x))$	$\exp(n\beta(x))$	$m-n=0$	$\dot{\varepsilon}_0 t\,(x - L/2 + m\phi)$

† Above $\beta(x) = x/L$. (E_0, ρ_0, c_0) correspond to properties at $x = 0$.
‡ α is an arbitrary real constant, $\psi = (1 + \alpha\beta(x))$ and $\phi = c_0^2\alpha t^2/(6L)$.

$\boxed{Q}$ **What are the forward and backward components?**

Decomposition of Displacement Field - I

Objective : To determine the forward and backward components of the displacement field.

- **Step 1:** Let $u_t = \hat{v}$ and $u_x = \hat{w}$ to get

$$\boldsymbol{U}_t + \boldsymbol{\mathcal{D}}(x,t)\boldsymbol{U}_x = \boldsymbol{F}(x,t)$$

where: $\boldsymbol{U} = \begin{Bmatrix} u_t \\ u_x \end{Bmatrix}$, $\boldsymbol{\mathcal{D}} = \begin{bmatrix} 0 & -c^2(x) \\ -1 & 0 \end{bmatrix}$, and $\boldsymbol{F} = \begin{Bmatrix} \frac{\hat{E}_x(x)}{\rho(x)}\hat{w} \\ 0 \end{Bmatrix}$

- **Step 2:** Use the linear transformation $\boldsymbol{U} = \boldsymbol{\mathcal{P}}(x,t)\boldsymbol{V}$ such that

$$u_t = v_1 + v_2, \quad v_{1,2} = \left(\frac{u_t \pm c(x)u_x}{2}\right) \quad \text{where:} \quad \boldsymbol{V} = \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

- **Step 3:** From $\boldsymbol{U} = \boldsymbol{\mathcal{P}}(x,t)\boldsymbol{V}$, we diagonalize $\boldsymbol{\mathcal{D}}(x,t)$ to get

$$\boldsymbol{V}_t + \begin{bmatrix} -c(x) & 0 \\ 0 & c(x) \end{bmatrix} \boldsymbol{V}_x = \boldsymbol{H}(x,t)$$

Decomposition of Displacement Field - I

Objective : To determine the forward and backward components of the displacement field.

- **Step 1:** Let $v_t = \hat{E}(x)u_x$ and $v_x = \rho(x)u_t$ to get

$$\boldsymbol{U}_t + \boldsymbol{\mathcal{D}}(x,t)\boldsymbol{U}_x = \boldsymbol{F}(x,t) \qquad \text{Non-unique !}$$

where: $\boldsymbol{U} = \begin{Bmatrix} u \\ v \end{Bmatrix}$, $\boldsymbol{\mathcal{D}} = \begin{bmatrix} 0 & -\frac{1}{\rho(x)} \\ -\hat{E}(x) & 0 \end{bmatrix}$, and $\boldsymbol{F} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

- **Step 2:** Use the linear transformation $\boldsymbol{U} = \boldsymbol{\mathcal{P}}(x,t)\boldsymbol{V}$ such that

$$u = v_1 + v_2, \quad v_{1,2} = \left(\frac{uZ(x) \pm v}{2Z(x)}\right) \quad \text{where:} \quad \boldsymbol{V} = \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

and $Z(x) = \sqrt{\hat{E}(x)\rho(x)}$

- **Step 3:** From $\boldsymbol{U} = \boldsymbol{\mathcal{P}}(x,t)\boldsymbol{V}$, we diagonalize $\boldsymbol{\mathcal{D}}(x,t)$ to get

$$\boldsymbol{V}_t + \begin{bmatrix} -c(x) & 0 \\ 0 & c(x) \end{bmatrix} \boldsymbol{V}_x = \boldsymbol{H}(x,t)$$

when $\boldsymbol{H}(x,t) \neq 0$, v_1 and v_2 are **Riemann Variables**.

Decomposition of Displacement Field - I

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- **Step 1:** Let $v_t = \hat{E}(x)u_x$ and $v_x = \rho(x)u_t$ to get

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where: $\boldsymbol{U} = \begin{Bmatrix} u \\ v \end{Bmatrix}$, $\boldsymbol{\mathcal{D}} = \begin{bmatrix} 0 & -\frac{1}{\rho(x)} \\ -\hat{E}(x) & 0 \end{bmatrix}$, and $\boldsymbol{F} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

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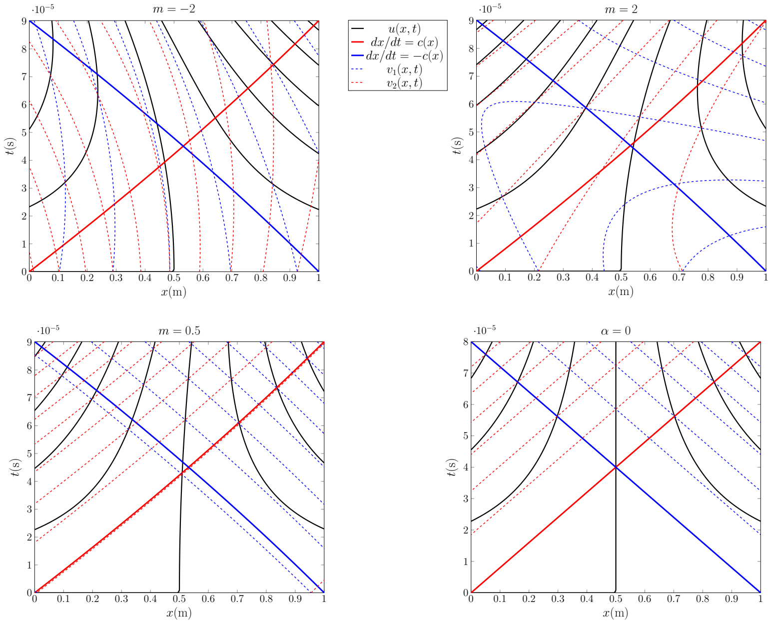
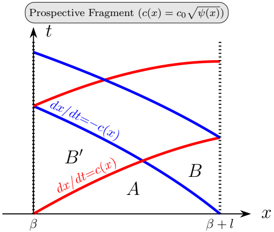
$$\boldsymbol{V}_t + \begin{bmatrix} -c(x) & 0 \\ 0 & c(x) \end{bmatrix} \boldsymbol{V}_x = \boldsymbol{H}(x,t)$$

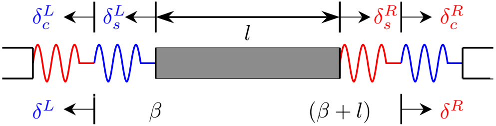
when $\boldsymbol{H}(x,t) = 0$, v_1 and v_2 are **Riemann Invariants**.

Decomposition of Displacement Field - II



Boundary Conditions





$$u(\beta, t) = \dot{\varepsilon}_0(\beta - L/2)t + \delta_c^L(t)$$
$$u(\beta + l, t) = \dot{\varepsilon}_0(\beta + l - L/2)t - \delta_c^R(t)$$

$$E(\beta)\, u_x(\beta, t) = \sigma_c(\delta_c(t))$$
$$E(\beta + l)\, u_x(\beta + l, t) = \sigma_c(\delta_c(t))$$

where $\delta_c^L(t) + \delta_c^R(t) = \delta_c(t)$ and $\sigma_c(\delta_c(t))$ is a suitable cohesive law.

ODE and Minimum fragment size $l_{\min}$

- From the boundary conditions and after non-dimensionalising:

$$\dot{\tilde{\delta}}_c(\tilde{t}) + \tilde{\delta}_c(\tilde{t}) \exp(1 - \tilde{\delta}_c(\tilde{t})) = \dot{\tilde{\varepsilon}}_0 \tilde{t} + \dot{\tilde{\varepsilon}}_0 \beta_1$$

where

$$\beta_1 = \frac{\tilde{l}(-3 + 2(m+1)}{4(1+m)} \left(1 + \frac{\epsilon^2 + C}{\epsilon(C+1)}\right), \ \epsilon = \frac{E(\tilde{\beta} + \tilde{l})}{E(\tilde{\beta})}, \ C = \frac{c(\tilde{\beta} + \tilde{l})}{c(\tilde{\beta})}$$

- We numerically solve this ODE to obtain values of $\tilde{\delta}_c(\tilde{t})$.
- Conditions for fragmentation to occur:

$$\delta_c(t_{\mathrm{cr}}) = \delta^* \implies \tilde{\delta}_c(\tilde{t}_{\mathrm{cr}}) = 1 \quad (1)$$

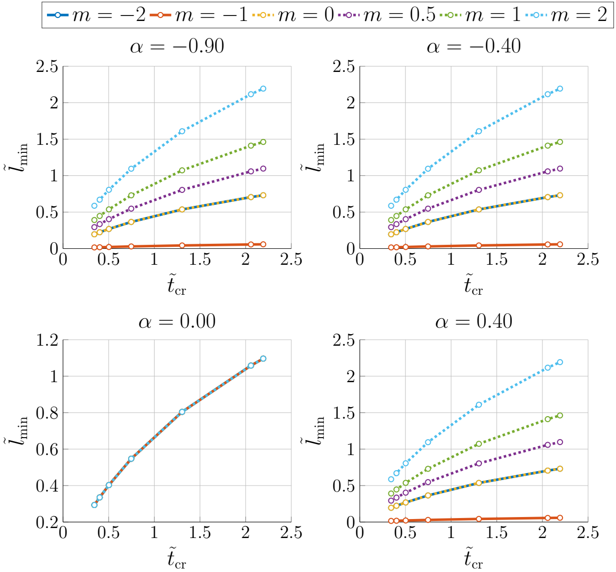
$$\dot{\delta}_s(t_{\mathrm{cr}}) = 0 \implies \dot{\delta}_c(t_{\mathrm{cr}}) = \dot{\varepsilon}_0 l_{\min} \implies \dot{\tilde{\delta}}_c(\tilde{t}_{\mathrm{cr}}) = \dot{\tilde{\varepsilon}}_0 \tilde{l}_{\min} \quad (2)$$

- The conditions are substituted in the ODE to obtain an implicit equation in terms of $\tilde{l}_{\min}$.
- This implicit equation is solved to obtain

$$\tilde{l}_{\min} = f(\alpha, m, \beta, L, \dot{\varepsilon}_0, \tilde{t}_{\mathrm{cr}})$$

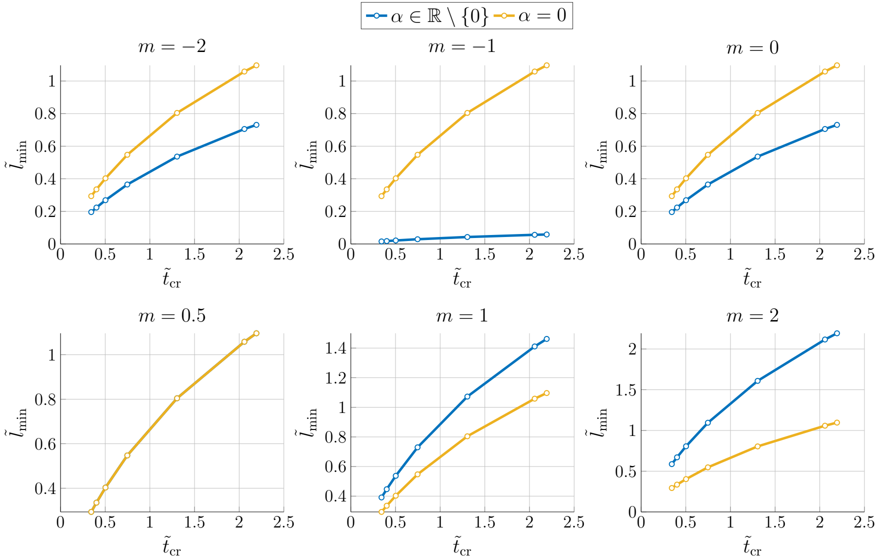
Results - I

Dependence on m:



Results - II

Dependence on α:



Open Questions

- Fragment size distribution
- Number of fragments
- Robust methods for decomposition of displacement fields even when the field is described in terms of special functions
- Fragmentation in higher dimensions
- Fragmentation under multi-axial loading

Thank you !

Questions ?