Hodge decomposition in variable exponent spaces with applications to regularity

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Based on join works with **Swarnendu Sil** and **Mikhail Surnachev**

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CHARLES UNIVERSITY

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Faculty of Mathematics

SFB 1283

Problems with differential forms

ω is a differential form; $d\omega$ is an exterior derivative ; $\delta\omega$ is a co differential.

Hodge Laplacian	$\Delta\omega = d\delta\omega + \delta d\omega = f$,
first order "div-curl" systems	$d\omega = f, \quad \delta\omega = g$,
Hodge-Dirac system	$D = d + \alpha\delta$, $\alpha \in \mathbb{R} \setminus \{0\}$,
"Bogovskii" type problems	$df = 0$.

Goals: solvability and regularity theory in spaces with variable exponent $L^{p(x)}(\Lambda M)$.
M is a compact n-dimential Riemanninan manifold with the boundary bM.

Goals for nonlinear problems:

1. Nonlinear problems with $p(x)$-Laplacian $\delta(|d\omega|^{p(x)-2}du) = \delta F$.
2. Finite element approximation for nonlinear problems with forms (curl-p-Laplacian).

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Regularity theory

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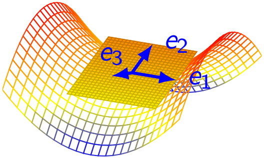
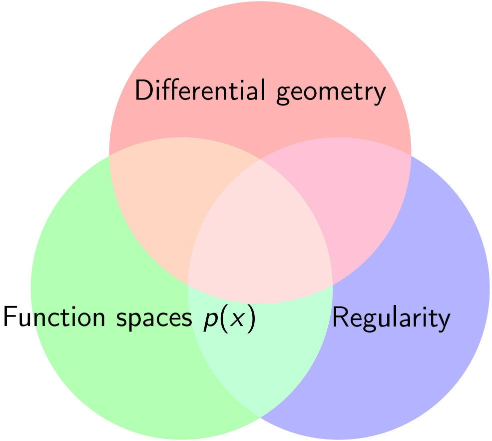


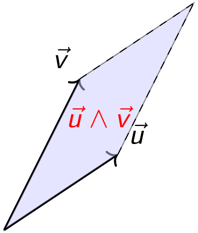


Problems

1. for non-standard spaces one has to develop theory from scratch;
2. no elliptic estimtes avaliable;
3. work on nonorientable manifold.

Vocabulary: vectors

Wedge product



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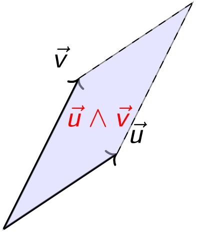

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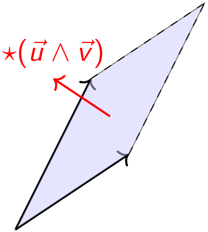
Hodge star



k-vector in $\mathbb{R}^n$
to $n - k$-vector

Sir W. V. D. Hodge

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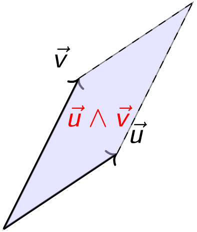

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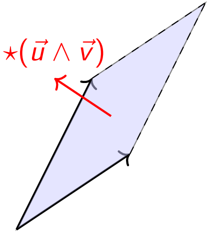

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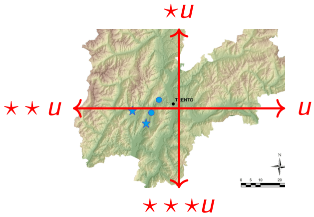
2d Hodge $\star$



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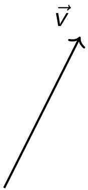
Covector: 1-form



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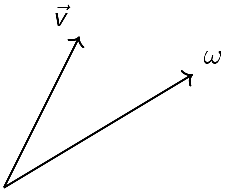

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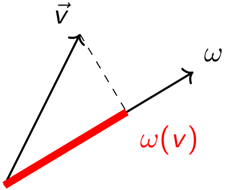

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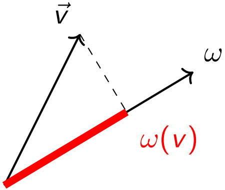

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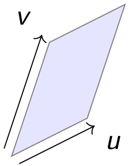

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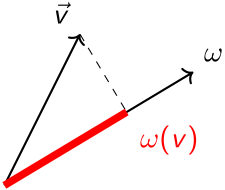

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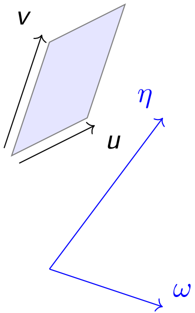

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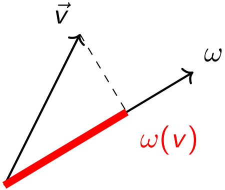

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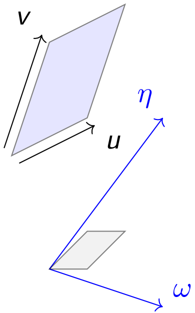

$$\omega \wedge \eta(u, v), \quad u \wedge u = 0.$$

The Notion of Differential Form

To define the differential forms and operations on them we need a lot of exterior algebra.

Naive definition

$n = 1$: the form is the object $\omega = f(x)dx$ in the integral $\int_a^b f(x)dx$ over the interval $[a, b]$.

$n \geq 1$ – the dim of the space and we also need to consider $0 \leq k \leq n$ – the dim of the path (oriented surface or manifold) we integrate over.

$k = 0$: scalar functions.

$k = 1$: integrate over oriented $1d$-oriented curve in $\mathbb{R}^n$. Could be described as vector fields .

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Operations with differential forms

Scalar functions	Differential forms
$(f, g) \to fg$	$\omega \wedge \eta = (-1)^{k\ell} \eta\omega$
$d(fg) = (df)g + f(dg)$	$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$.

$d(d\omega) = 0$ The operation $\omega \to d\omega$

Order of ω	$d\omega =$ "usual operator"
$k = 0$	$f \to \nabla f$
$k = 1$	$f \to \operatorname{curl} f$
$k = 2$	$f \to \operatorname{div} f$

Even and odd forms

We work on sufficiently **not necessary orientable** smooth Riemanninan manifold.

The Boy Surface,
at Oberwolfach

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So we need to consider even and odd forms. Let $x = x(y)$ coordinate change and $J = \frac{\partial x}{\partial y}$.
We have before change of coordinates:

$$\sum_K \omega_K(x) dx^K$$

after

$$\sum_I \omega'_I(y) dy^I$$

$$\textbf{even} \quad \omega'_I(y) = \sum \omega_k(x(y)) \frac{\partial x^k}{\partial y^I}.$$

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Bundle of differential forms

Following approach of De Rham: work with the totality ΩM of differential forms of all degrees $0, \ldots, \dim M$ and parities (odd/even).
On an n-dimensional manifold each element of ΩM can be decomposed into $2(n+1)$ homogeneous forms:

$$f = \sum_{r=0}^{n} f_e^r + \sum_{r=0}^{n} f_o^r, \quad \deg f_e^k, f_o^k = k, \quad k = 0, \ldots, n$$

the forms f_e^k even and f_o^k are odd.

Example

$$\omega = 1 + xdy + zdxdy.$$

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Properties of bundle

We set

$$\langle f, g\rangle = \sum_{r=0}^{n} \langle f_e^r, g_e^r\rangle + \sum_{r=0}^{n} \langle \varepsilon f_o^r, \varepsilon g_o^r\rangle,$$

where ε is an odd form of degree zero with values ± 1 in any coordinate system ("orientation"). On r-forms the scalar product $\langle \cdot, \cdot\rangle$ is defined in the standard way:

$$(\omega, \eta) = \sum_{I} \omega_I \eta^I = \sum_{IK} G^{IK} \omega_I \eta_K$$

where the summation is over ordered sets $I, K \in \mathcal{I}(r)$, $\eta^I = g^{i_1 j_1} \dots g^{i_r j_r} \eta_{j_1 \dots j_r}$ with the summation over all r-tuples $(j_1, \dots, j_r)$, and G^{IK} is the determinant of the matrix at the intersection of rows I and columns K of the matrix $\{g^{ij}\}$. We denote $|\omega| = \sqrt{\langle \omega, \omega\rangle}$.

Generalized differential and codifferential

For k-forms f and g of the same parity we denote

$$(f,g) = \int_M \langle f,g\rangle \, dV = \int_M f \wedge *g.$$

$\omega \in L^1_{loc}(M,\Lambda)$ has differential $d\omega = f \in L^1_{loc}(M,\Lambda)$ if for any $\varphi \in C^1_o(M,\Lambda)$

$$(\omega, \delta\varphi)_M = (f,\varphi)_M.$$

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Admissible coordinate system

An admissible boundary coordinate system on a manifold M with boundary bM of class $C^{s,\mu}$ is a coordinate system of class $C^{s,\mu}$: interior points of boundary coordinate neighbourhood are mapped into $x^n > 0$, image of bM belongs to $x^n = 0$, and metric is

$$ds^2 = \sum_{\gamma,\delta=1}^{p-1} g_{\gamma,\delta}(x'_n, 0) dx^\gamma dx^\delta + (dx^n)^2 \quad \text{on } \sigma.$$

Boundary values and the normal and tangent components are defined via admissible boundary coordinate system.
Normal in admissible coordinate system

$$\nu = -dx^n.$$

Normal and tangential part

Components in admissible coordinate system

$$\omega = \sum_{n \notin I} \omega_I dx^I + \sum_{n \in I} \omega_I dx^I$$

$$\omega_I \begin{cases} n \notin I \rightarrow \text{tangential part;} \\ n \in I \rightarrow \text{normal part.} \end{cases} ,$$

On the boundary form decompose

$$\omega = t\omega + n\omega.$$

$$t\omega = 0 \Leftrightarrow \nu \wedge \omega = 0,$$
$$n\omega = 0 \Leftrightarrow \nu \lrcorner \omega = 0.$$

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Harmonic fields

Let M be of the class $C^{s+2,1}$, $s \in \{0\} \cup \mathbb{N}$, and let $n = \dim M$.
Introduce the spaces of even and odd harmonic fields (i.e. $W^{1,2}(M,\Lambda)$ forms with zero differential and codifferential) of degree r with vanishing tangential part on the boundary $\mathcal{H}_T(M,\Lambda^r_*)$, $* \in \{e,o\}$, the spaces of even and odd harmonic forms of degree r with vanishing normal part on the boundary $\mathcal{H}_N(M,\Lambda^r_*)$, $* \in \{e,o\}$. Then let

$$\mathcal{H}_T(M,\Lambda_*) = \bigoplus_{r=0}^{\dim M} \mathcal{H}_T(M,\Lambda^r_*), \quad * \in \{e,o\},$$

$$\mathcal{H}_N(M,\Lambda_*) = \bigoplus_{r=0}^{\dim M} \mathcal{H}_N(M,\Lambda^r_*), \quad * \in \{e,o\},$$

$$\mathcal{H}_T(M) = \mathcal{H}_T(M,\Lambda_e) \bigoplus \mathcal{H}_T(M,\Lambda_o),$$

$$\mathcal{H}_N(M) = \mathcal{H}_N(M,\Lambda_e) \bigoplus \mathcal{H}_N(M,\Lambda_o).$$

Connection to the geometry

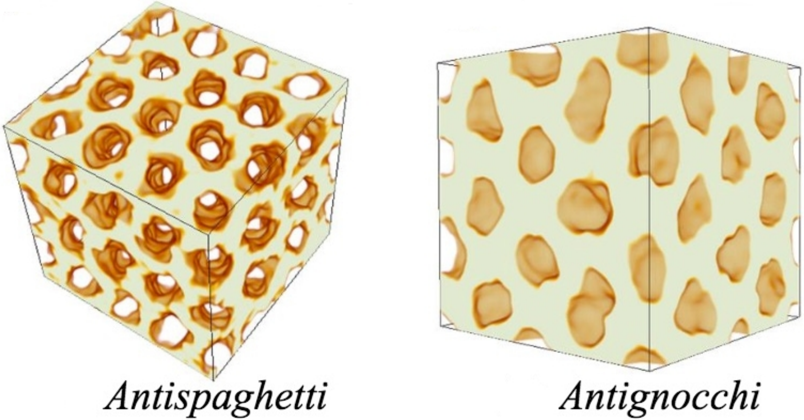
The dimensions of kernels satisfy:

$$\dim \mathcal{H}_T(\Lambda^r) = B_{n-r}, \quad \dim \mathcal{H}_N(\Lambda^r) = B_r,$$

where B_r is the r-th Betti number of M (rank of the r-th homology group of M), the number B_0 represents the number of connected components of M, $B_n = 0$, and if M is contractible then $B_r = 0$ for $r = 1, \dots, n-1$.

Nuclear pasta

Betti number of M (number is the number of r-dimensional holes)



Picture from Wikipedia Nuclear pasta

4 classical BVP for the Hodge Laplacian

$$\Delta u = f.$$

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Hodge Laplacian with these boundary condition is formally symmetric.

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Variational problems

$$\mathcal{F}[\omega] = \frac{1}{2}(d\omega, d\omega) + \frac{1}{2}(\delta\omega, \delta\omega) - (\varphi, d\omega) - (\psi, \delta\omega) - (\eta, \omega) \to \min \text{ over } X,$$

$\eta \in L^2$, $\varphi, \psi \in W^{1,2}$.

- $X = W_0^{1,2}(M, \Lambda)$, $\omega \in W^{2,2}(M, \Lambda)$ and satisfies

$$\triangle\omega = \eta + \delta\varphi + d\psi \quad \text{a.e. in } M, \quad t\omega = 0, \quad n\omega = 0.$$

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Variational problems

$$\mathcal{F}[\omega] = \frac{1}{2}(d\omega, d\omega) + \frac{1}{2}(\delta\omega, \delta\omega) - (\varphi, d\omega) - (\psi, \delta\omega) - (\eta, \omega) \to \min \text{ over } X,$$

$\eta \in L^2$, $\varphi, \psi \in W^{1,2}$.

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History

Linear theory: Hodge, De Rham, Kodaira, Duff, Spencer; Morrey later extended by R. Kress, Schwarz, Bolik, Mitrea(s)

Nonlinear theory :	Uhlenbeck; Hamburger
Sobolev spaces of differential forms Nonlinear Hodge Theory :	Iwaniec, Scott, Strofollini '99
Modern development :	Csató, Dacorogna, Kneuss
Partial regularity:	Beck and Stroffolini '13
Variational method for p:	Sil '16,'19,'21
Calderon-Zygmund estimates :	Lee, Ok, Pyo '24
Generalized Sobolev-Orlicz spaces Lavrentiev gap:	Balci, Surnachev '24

Spaces with variable exponent

$\omega \in L^1_{loc}$, $d\omega = f \in L^1_{loc}$ and $t\omega = 0$ if

$$(\omega, \delta\varphi) = (f, \varphi) \quad \forall \varphi \in C^1.$$

$$\|\omega\|_{L^{p(\cdot)}(M,\Lambda)} = \inf\{\lambda > 0 \,:\, \|(|\omega|\lambda^{-1})^{p(\cdot)}\|_{L^1(M)} < \infty\}, \quad * \in \{e, o\}.$$

Let M be at least $C^{s,1}$, $s \in \mathbb{N}$ and $(U_\alpha, \varphi_\alpha)$ be a finite atlas of M. On $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ we denote $p_\alpha(\cdot) = p(\varphi_\alpha^{-1}(\cdot))$.

We assume that p has the logarithmic modulus of continuity:

$$|p(x) - p(y)| \leq \frac{L}{\log\big(e + (\mathrm{dist}\,(x,y))^{-1}\big)}, \quad x, y \in M \quad \text{Zhikov '90s}$$

In each coordinate system components belong to $W^{s,p_\alpha(\cdot)}(\varphi_\alpha(U_\alpha))$. The norm could be expressed in terms of covariant derivatives:

$$\sum_{l=0}^{s} \|\,|\nabla^l \omega_e|\,\|_{L^{p(\cdot)}(M,\Lambda)} + \sum_{l=0}^{s} \|\,|\nabla^l (\varepsilon\omega_o)|\,\|_{L^{p(\cdot)}(M,\Lambda)}.$$

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$$W^{\delta,q}(M,\Lambda) = \{\omega \in L^q(M,\Lambda) \ : \ \delta\omega \in L^q(M,\Lambda)\},$$
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here is understood in the sense

$$(\omega, d\varphi) = (d\omega, \varphi)$$

for all $\varphi \in C^1(M)$.

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Lavrentiev Phenomenon and Density

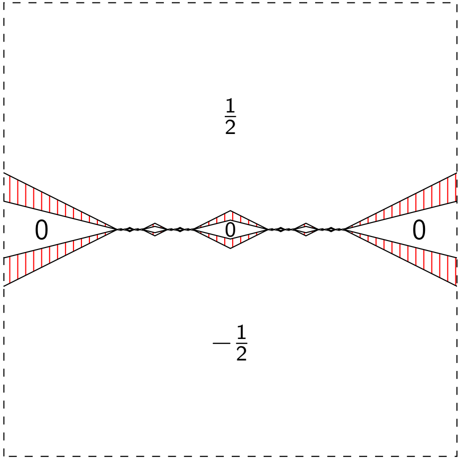
Balci and Surnachev [CalcVar2024]:

1. For generalized Sobolev-Orlicz spaces of differential forms $W^{d,\varphi(\cdot)}(\Omega;\Lambda^k)$ we construct the examples on Lavrentiev gap and obtain nondensity result using fractals

$$H^{d,p(\cdot)}(\Omega,\Lambda) \neq W^{d,p(\cdot)}(\Omega,\Lambda).$$

2. For the space $W^{s,p(\cdot)}(\Omega,\Lambda)$ density is provided if $p(x)$ is log-Holder continuous.
3. Construction of fractal barriers is similar to Balci, Diening, Surnachev [CalcVar2021].

Function u



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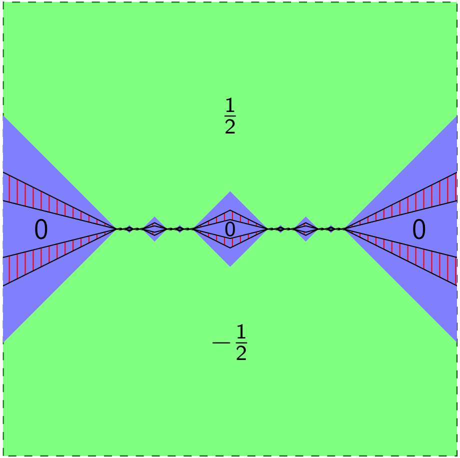
Function u and exponent p



Dirichlet problem for Hodge Laplacian in Variable exponent spaces

Theorem [Balci, Sil, Surnachev]

Let $\eta \in W^{s,p(\cdot)}(M,\Lambda)$, $\varphi \in W^{s+2,p(\cdot)}(M,\Lambda)$, and $\psi \in W^{s+1,p(\cdot)}(M,\Lambda)$. Let $(\eta, h_T) = [\psi, h_T]$ for all $h_T \in \mathcal{H}_T(M)$. Then there exists a solution $\omega \in W^{s+2,p(\cdot)}(M,\Lambda)$ of the boundary value problem

$$\triangle\omega = \eta, \quad t\omega = t\varphi, \quad t\delta\omega = t\psi,$$

such that

$$\|\omega\|_{W^{s+2,p(\cdot)}(M,\Lambda)} \leq C(\|\eta\|_{W^{s,p(\cdot)}(M,\Lambda)} + \|\varphi\|_{W^{s+2,p(\cdot)}(M,\Lambda)} + \|\psi\|_{W^{s+1,p(\cdot)}(M,\Lambda)})$$

where $C = C(p_-, p_+, c_{\log}(p), M, s)$.

$$[f,g] = \int_{bM} \langle \nu \wedge f, g\rangle d\sigma = \int_{bM} \langle f, \nu \lrcorner g\rangle d\sigma = \int_{bM} f \wedge *g.$$

Hodge decomposition

Let $\omega \in W^{s,p(\cdot)}(M,\Lambda)$. Then there exist $\alpha, \beta \in W^{s+1,p(\cdot)}(M,\Lambda)$ and $h \in \mathcal{H}_T(M)$ such that

$$\omega = h + d\alpha + \delta\beta,$$
$$t\alpha = 0, \quad \delta\alpha = 0, \quad t\beta = 0, \quad d\beta = 0,$$
$$\|\alpha\|_{W^{s+1,p(\cdot)}(M,\Lambda)}, \|\beta\|_{W^{s+1,p(\cdot)}(M,\Lambda)} \leq C\|\omega\|_{W^{s,p(\cdot)}(M,\Lambda)}.$$

Let $\omega \in W^{s,p(\cdot)}(M,\Lambda)$, $s \in \mathbb{N} \cup \{0\}$. Then there exist $\alpha, \beta \in W^{s+1,p(\cdot)}(M,\Lambda)$ and $h \in \mathcal{H}_N(M)$ such that

$$\omega = h + d\alpha + \delta\beta,$$
$$n\alpha = 0, \quad \delta\alpha = 0, \quad n\beta = 0, \quad d\beta = 0,$$
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Methods

Diening and Růžička first results from 2003 Calderon-Zygmund operators in variable exponent spaces; estimates in half space.
Diening, Harjulehto, Hästö, and Růžička 2011, Section 6.3
Diening and Růžička 2003, second order estimates for classical Laplacian in variable exponent spaces. Estimates for potentials in $p(x)$ spaces+ Morrey approach
We assume that $p : M \to [p_-, p_+]$

$$|p(x) - p(y)| \leq \frac{C}{\log \frac{1}{\mathrm{dist}(x,y)+e}}.$$

Methods are classical and contained in the book by Charles B. Morrey

Variable exponent problem with differential forms

$$d^{*}\left(a(x)\,|du|^{p(x)-2}\,du\right) = d^{*}F \qquad \text{in } \Omega.$$

$u \in W^{1,1}\left(\Omega; \Lambda^{k}\right)$ is called a **weak solution** if $u \in W^{d,p(\cdot)}\left(\Omega; \Lambda^{k}\right)$ and satisfies

$$\int_{\Omega} \left\langle a(x)\,|du|^{(p(x)-2)}\,du, d\varphi \right\rangle = \int_{\Omega} \langle F, d\varphi \rangle \qquad \text{for every } \varphi \in W_{T}^{d,p(\cdot)}\left(\Omega; \Lambda^{k}\right).$$

Higher intergrability and Hölder continuity

For $k = 0$ regularity goes back to Acerbi and Mingione.

Balci, Sil, Surnachev 2024

Let $n \geq 2$, $N \geq 1$ and $0 \leq k \leq n-1$ be integers and let $\Omega \subset \mathbb{R}^n$ be open, bounded subset with smooth boundary. Let $p : \Omega \to [\gamma_1, \gamma_2]$ be log-Holder with $1 < \gamma_1 \leq \gamma_2 < \infty$, and we set

$$\lim_{R \to 0} \omega(R) \log\left(\frac{1}{R}\right) := L_1 < +\infty.$$

Let $u \in W^{d,p(\cdot)}_{\mathrm{loc}}(\Omega; \Lambda^k)$ be a local weak solution to the system. Then

$$\left(\int_{B_{R/2}} |du|^{p(x)(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \leq$$

$$c \left(\int_{B_R} |du|^{p(x)} \, dx + 1 \right) + c \left(1 + \int_{B_R} |F - \xi|^{\frac{\gamma_1(1+\sigma)}{\gamma_1 - 1}} \, dx \right)^{\frac{1}{1+\sigma}}.$$

FEM for curl-p Laplacian

System arising from applied superconductivity [Wan and Laforest 2020, SIAM J. NUMER. ANAL.]

$$\operatorname{curl}(|\operatorname{curl} u|^{p-2} \operatorname{curl} u) = f,$$
$$\operatorname{div}(u) = 0.$$

This is the Euler-Lagrange equation for the variational problem

$$\int_\Omega \frac{|\operatorname{curl} u|^p}{p} - fu \to \min$$

The corresponding energy space is $W^{1,p}(\operatorname{curl})$ with $\operatorname{div} u = 0$.
[Balci, Kaltenbach in progress] Under realistic regularity assumptions we derive optimal error estimates in turms of natural distance. These estimates depend on the existence of an stable interpolation operator of Schöberl type.

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Summary

We study linear and nonlinear problems with differential forms.
We obtain

1. Generalised spaces with differential forms on smooth manifolds;
2. Solvability and estimates for different boundary valiue problems for Hodge Laplacian in variable exponenet spaces; Hodge decomposition;
3. Solvability of first-order systems and Gaffney's inequality;
4. Solvability for Hodge-Dirac system and non-elliptic first order systems;
5. Higher intergrability and Hölder continuity for the variable exponent p-Laplacian for differential forms;
6. FEM for nonlinear problems;

These results obtained together with Swarnendu Sil and Mikhail Surnachev. And work in progress with Alex Kaltenbach.