Analysis of viscoelastic fluids

Stability near equilibrium

Michal Bathory

Faculty of Mathematics and Physics
Charles University

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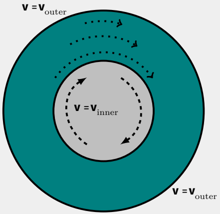
- Stability of a (steady) flow ⇔ its resistance to **finite** perturbations.
- The flow is **induced by the movement of a solid boundary**.
- Let the fluid be viscous $\nu > 0$ and incompressible div $\mathbf{v} = 0$.
- Well known example of this setting is the Couette experiment with a fluid between two rotating cylinders.



- The basic axisymmetric steady state given by

$$\mathbf{v}_\theta(r) = C_1 r + C_2 r^{-1}$$

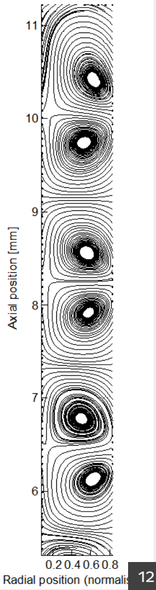
is stable if $\mathbf{v}_{outer}$ is not much smaller than $\mathbf{v}_{inner}$ and if the annulus is not too thin.

Taylor vortices

- In 1922, Sir G. I. Taylor described necessary & sufficient criteria for the transition of the basic flow to the vortex flow.
- Assuming that the disturbance is symmetric ($\partial_\theta \equiv 0$), he obtained the explicit solutions in terms of the Fourier series of the Bessel functions:

$$f(r) = \sum_{s=1}^{\infty} \alpha_s[f]B(k_s r)$$

- Impressive, but probably useless in more general settings, such as
  - Complex fluids
  - Irregular geometries
- Our aim is to provide just sufficient criteria for stability, but for **viscoelastic fluids** and in **general domains**.



Navier-Stokes case

Suppose that $\mathbf{v}$ (and p) is a solution of

$$\begin{aligned} \operatorname{div}\mathbf{v} &= 0 \\ \partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= 0, \qquad \nu > 0 \\ \mathbf{v}\,|_{\partial\Omega} &= \mathbf{w}_D \qquad (\mathbf{w}_D\cdot\mathbf{n} = 0) \end{aligned}$$

corresponding to some initial datum $\mathbf{v}(0) = \mathbf{v}_0$, and let $\mathbf{u}$ be the steady state solution of the same system. Subtracting the equations and testing with the difference $\mathbf{v} - \mathbf{u}$ leads to

$$\frac{1}{2}\partial_t \int_\Omega |\mathbf{v}-\mathbf{u}|^2 + \nu \int_\Omega |\nabla(\mathbf{v}-\mathbf{u})|^2 = -\int_\Omega (\mathbf{v}-\mathbf{u})\cdot\nabla\mathbf{u}\cdot(\mathbf{v}-\mathbf{u}) \leq \|\nabla\mathbf{u}\|_\infty \|\mathbf{v}-\mathbf{u}\|_2^2$$

We observe:

- Exponential stability if $\|\nabla\mathbf{u}\|_\infty$ is sufficiently small (depending on ν and Poincaré constant of Ω).
- Does not need $\partial_t\mathbf{u} = 0$.
- Works also for weak solutions, provided they satisfy the energy inequality.

Equations for viscoelastic fluids

The momentum equation gets an additional term:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = 2\,\mathrm{div}(\mathbb{B})$$

and the unknown elastic stress tensor $\mathbb{B}$ solves the Oldroyd-B/Giesekus equation

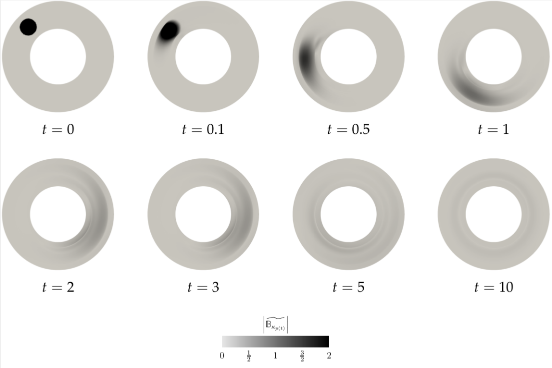
$$\partial_t \mathbb{B} + \mathbf{v} \cdot \nabla \mathbb{B} + \delta_1(\mathbb{B} - \mathbb{I}) + \delta_2(\mathbb{B}^2 - \mathbb{B}) - \lambda \Delta \mathbb{B} = \nabla \mathbf{v} \mathbb{B} + \mathbb{B} \nabla^T \mathbf{v}, \qquad \delta_1, \delta_2 > 0, \qquad \lambda \geq 0$$

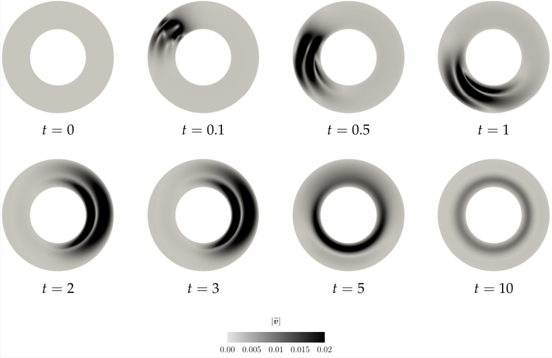
$$\lambda \mathbf{n} \cdot \nabla \mathbb{B} \,|_{\partial\Omega} = 0$$

- The δ_1, δ_2-terms model an elastic damping.
- The stress diffusion term is optional. It simplifies the analysis, but the corresponding equation for $\mathbb{B}^{-1}$ is “lost”. Both models $\lambda = 0$ and $\lambda > 0$ seem physically relevant.
- Existence of global-in-time, three-dimensional solution for this system is known if $\delta_1 = 0$ and $\lambda = 0$, see [Bulíček, Málek, Los; 2024].
- One has to ensure that the matrix $\mathbb{B}$ is **positive definite**.

Couette for viscoelastic flows

- Allowing the fluid to store elastic energy makes the Couette experiment even more interesting.
- The initial perturbation can now be encoded not only in $\mathbf{v}(0)$, but also in $\mathbb{B}(0)$.
- Nice illustrations can be found in [Dostalík, Průša, Tůma; 2019]:





The Lyapunov functional

- We need a way to measure distance of two solutions, say $(\mathbf{v}, \mathbb{B})$ and $(\mathbf{u}, \mathbb{A})$.
- The naive guess

$$L_{naive} = \int_\Omega |\mathbf{v} - \mathbf{u}|^2 + \int_\Omega |\mathbb{B} - \mathbb{A}|^2$$

does not work since in its time derivative, cubic terms like $(\nabla \mathbf{v} - \nabla \mathbf{u})(\mathbb{B} - \mathbb{A})^2$ coming from the objective derivative will spoil the estimate.

- A more natural candidate is

$$L = \frac{1}{2} \int_\Omega |\mathbf{v} - \mathbf{u}|^2 + \int_\Omega \psi(\mathbb{B}\mathbb{A}^{-1}),$$

where ψ is the free (elastic) energy function

$$\psi(\mathbb{Y}) = \mathrm{tr}(\mathbb{Y} - \mathbb{I}) - \ln \det \mathbb{Y}, \qquad \mathbb{Y} > 0.$$

Function ψ is **convex, non-negative and** $\psi(\mathbb{I}) = 0$.

- The correct "testing procedure" follows from the form of the time derivative

$$\boxed{\partial_t \psi(\mathbb{B}\mathbb{A}^{-1}) = \partial_t \mathbb{B} \cdot (\mathbb{A}^{-1} - \mathbb{B}^{-1}) + \partial_t \mathbb{A} \cdot (\mathbb{A}^{-1} - \mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1})}$$

Stability of smooth solutions

This is the approach taken in [Dostalík, Průša, Tůma; 2019] and it leads to the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_\Omega\left(\frac{1}{2}|\mathbf{v}-\mathbf{u}|^2+\psi(\mathbb{A}^{-1}\mathbb{B})\right)}_{L}+\nu\int_\Omega|\nabla(\mathbf{v}-\mathbf{u})|^2+\delta_1\int_\Omega|\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}}-\mathbb{B}^{-\frac{1}{2}}|^2+\delta_2\int_\Omega|\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}}|^2$$

$$=-\int_\Omega(\mathbf{v}-\mathbf{u})\cdot\underbrace{\nabla\mathbf{u}}_{\text{small}}\cdot(\mathbf{v}-\mathbf{u})+2\int_\Omega(\mathbf{v}-\mathbf{u})\cdot\underbrace{\nabla\mathbb{A}^{-\frac{1}{2}}}_{\text{small}}\cdot(\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}})+2\int_\Omega\nabla(\mathbf{v}-\mathbf{u})\cdot\underbrace{(\mathbb{A}^{-\frac{1}{2}}-\mathbb{A}^{\frac{1}{2}})}_{\text{small}}(\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}})$$

- **Conclusion:** if the steady solution $(\mathbf{u},\mathbb{A})$ is such that $\|\nabla\mathbf{u}\|_\infty$, $\|\mathbb{A}-\mathbb{I}\|_\infty$ and $\|\nabla\mathbb{A}\|_\infty$ are sufficiently small, then
$$\frac{\mathrm{d}}{\mathrm{d}t}L\leq 0.$$
- However, this was done under additional assumptions:
  - $\partial_t\mathbf{u}=0$, $\partial_t\mathbb{A}=0$,
  - $\lambda=0$,
  - **The perturbed solution** $(\mathbf{v},\mathbb{B})$ **is smooth.**
- We remove all these assumptions and also show that L decays **exponentially** fast.

Stress-diffusive case $\lambda > 0$

To handle the stress-diffusion term $-\lambda\Delta\mathbb{B}$, we derive the following identity

$$
\begin{aligned}
&\nabla\mathbb{B}\cdot\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})+\nabla\mathbb{A}\cdot\nabla(\mathbb{A}^{-1}-\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1})\\
&\quad=\nabla\mathbb{B}\cdot\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})-\nabla\mathbb{A}\cdot\mathbb{A}^{-1}\nabla\mathbb{B}\mathbb{A}^{-1}-2\nabla\mathbb{A}\cdot\nabla\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}+\nabla\mathbb{A}\cdot\nabla\mathbb{A}^{-1}\\
&\quad=|\mathbb{B}^{\frac12}\nabla\mathbb{B}^{-1}\mathbb{B}^{\frac12}|^2-2\mathbb{B}^{\frac12}\nabla\mathbb{B}^{-1}\mathbb{B}^{\frac12}\cdot\mathbb{B}^{\frac12}\nabla\mathbb{A}^{-1}\mathbb{B}^{\frac12}+2\mathbb{A}\nabla\mathbb{A}^{-1}\cdot\nabla\mathbb{A}^{-1}\mathbb{B}+\nabla\mathbb{A}\cdot\nabla\mathbb{A}^{-1}\\
&\quad=|\mathbb{B}^{\frac12}\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})\mathbb{B}^{\frac12}|^2-\mathbb{B}\nabla\mathbb{A}^{-1}\cdot\nabla\mathbb{A}^{-1}\mathbb{B}+2\mathbb{A}\nabla\mathbb{A}^{-1}\cdot\nabla\mathbb{A}^{-1}\mathbb{B}-\mathbb{A}\nabla\mathbb{A}^{-1}\cdot\nabla\mathbb{A}^{-1}\mathbb{A}\\
&\quad=|\mathbb{B}^{\frac12}\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})\mathbb{B}^{\frac12}|^2+\nabla\mathbb{A}^{-1}\mathbb{B}\cdot(\mathbb{A}-\mathbb{B})\nabla\mathbb{A}^{-1}+\mathbb{A}\nabla\mathbb{A}^{-1}\cdot\nabla\mathbb{A}^{-1}(\mathbb{B}-\mathbb{A})\\
&\quad=|\mathbb{B}^{\frac12}\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})\mathbb{B}^{\frac12}|^2-\nabla\mathbb{A}^{-1}(\mathbb{B}-\mathbb{A})\cdot(\mathbb{B}-\mathbb{A})\nabla\mathbb{A}^{-1}\\
&\quad=|\mathbb{B}^{\frac12}\nabla(\mathbb{A}^{-1}-\mathbb{B}^{-1})\mathbb{B}^{\frac12}|^2-\underbrace{\nabla\mathbb{A}^{-1}\mathbb{A}^{\frac12}}_{\text{small}}\underbrace{(\mathbb{A}^{-\frac12}\mathbb{B}-\mathbb{A}^{\frac12})\cdot(\mathbb{B}\mathbb{A}^{-\frac12}-\mathbb{A}^{\frac12})}_{\text{Giesekus }\delta_2\text{ term}}\underbrace{\mathbb{A}^{\frac12}\nabla\mathbb{A}^{-1}}_{\text{small}}.
\end{aligned}
$$

Using the smallness assumptions on $(\mathbf{u}, \mathbb{A})$, we can arrive at the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_\Omega (\frac{1}{2}|\mathbf{v}-\mathbf{u}|^2 + \psi(\mathbb{A}^{-1}\mathbb{B}))}_{L} + \frac{\nu}{2}\int_\Omega |\nabla(\mathbf{v}-\mathbf{u})|^2 + \delta_1 \int_\Omega |\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}|^2 + \frac{\delta_2}{2}\int_\Omega |\mathbb{A}^{-\frac{1}{2}}\mathbb{B} - \mathbb{A}^{\frac{1}{2}}|^2 \leq 0.$$

We observe that

$$\begin{aligned}
\psi(\mathbb{A}^{-1}\mathbb{B}) &\leq \psi(\mathbb{A}^{-1}\mathbb{B}) + \psi(\mathbb{B}^{-1}\mathbb{A}) \\
&= \mathrm{tr}(\mathbb{A}^{-1}\mathbb{B} - \mathbb{I}) - \ln\det(\mathbb{A}^{-1}\mathbb{B}) + \mathrm{tr}(\mathbb{B}^{-1}\mathbb{A} - \mathbb{I}) - \ln\det(\mathbb{B}^{-1}\mathbb{A}) \\
&= (\mathbb{B} - \mathbb{A}) \cdot (\mathbb{A}^{-1} - \mathbb{B}^{-1}) = (\mathbb{B}^{\frac{1}{2}} - \mathbb{A}\mathbb{B}^{-\frac{1}{2}}) \cdot (\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}) = \mathbb{A}(\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}) \cdot (\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}) \\
&\leq |\mathbb{A}||\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}|^2
\end{aligned}$$

Therefore, if $\delta_1 > 0$, then there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} L + \varepsilon L \leq 0,$$

leading to the exponential decay of L.

We want to prove

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}|\mathbf{v}-\mathbf{u}|^2+\psi(\mathbb{A}^{-1}\mathbb{B})\right)+\nu\int_\Omega|\nabla(\mathbf{v}-\mathbf{u})|^2+\delta_1\int_\Omega|\mathbb{A}^{-1}\mathbb{B}^{\frac{1}{2}}-\mathbb{B}^{-\frac{1}{2}}|^2+\delta_2\int_\Omega|\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}}|^2$$
$$\leq-\int_\Omega(\mathbf{v}-\mathbf{u})\cdot\nabla\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})+2\int_\Omega(\mathbf{v}-\mathbf{u})\cdot\nabla\mathbb{A}^{-\frac{1}{2}}\cdot(\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}})+2\int_\Omega\nabla(\mathbf{v}-\mathbf{u})\cdot(\mathbb{A}^{-\frac{1}{2}}-\mathbb{A}^{\frac{1}{2}})(\mathbb{A}^{-\frac{1}{2}}\mathbb{B}-\mathbb{A}^{\frac{1}{2}})$$

for weak solutions, but the standard energy estimates only contain $\partial_t\psi(\mathbb{B})$ and $\partial_t\psi(\mathbb{A})$ (or $\partial_t\psi(\mathbb{A}^{-1})$).

Hence, we would really like to replace $(\mathbf{u},\mathbb{A})$ by some generic (smooth) test function $(\mathbf{w},\mathbb{Y})$.

Can we do that?

Yes, we **can** construct a weak solution such that, **for all** smooth **w** and $\mathbb{Y}$, there holds

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\mathbf{v}-\mathbf{w}|^2 + \psi(\mathbb{Y}^{-1}\mathbb{B})\right) + \nu\int_\Omega |\nabla(\mathbf{v}-\mathbf{w})|^2 + \delta_1\int_\Omega |\mathbb{Y}^{-1}\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}|^2 + \delta_2\int_\Omega |\mathbb{Y}^{-\frac{1}{2}}\mathbb{B} - \mathbb{Y}^{\frac{1}{2}}|^2$$

$$\le -\int_\Omega (\mathbf{v}-\mathbf{w})\cdot\nabla\mathbf{w}\cdot(\mathbf{v}-\mathbf{w}) + 2\int_\Omega (\mathbf{v}-\mathbf{w})\cdot\nabla\mathbb{Y}^{-\frac{1}{2}}\cdot(\mathbb{Y}^{-\frac{1}{2}}\mathbb{B} - \mathbb{Y}^{\frac{1}{2}}) + 2\int_\Omega \nabla(\mathbf{v}-\mathbf{w})\cdot(\mathbb{Y}^{-\frac{1}{2}} - \mathbb{Y}^{\frac{1}{2}})(\mathbb{Y}^{-\frac{1}{2}}\mathbb{B} - \mathbb{Y}^{\frac{1}{2}})$$

$$-\int_\Omega (\partial_t\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} - \nu\Delta\mathbf{w} - 2\,\mathrm{div}\,\mathbb{Y})\cdot(\mathbf{v}-\mathbf{w})$$

$$+\int_\Omega (\partial_t\mathbb{Y} + \mathbf{w}\cdot\nabla\mathbb{Y} + \delta_1(\mathbb{Y}-\mathbb{I}) + \delta_2(\mathbb{Y}^2 - \mathbb{Y}) - \nabla\mathbf{w}\mathbb{Y} - \mathbb{Y}(\nabla\mathbf{w})^T)\cdot(\mathbb{Y}^{-1} - \mathbb{Y}^{-1}\mathbb{B}\mathbb{Y}^{-1})$$

- The choice $\mathbf{w} = \mathbf{w}_D$ and $\mathbb{Y} = \mathbb{I}$ recovers the standard energy inequality for $(\mathbf{v}, \mathbb{B})$.
- The choice $\mathbf{w} = \mathbf{u}$ and $\mathbb{Y} = \mathbb{A}$ gives the stability (or uniqueness) result.
- The choice $\mathbf{w} = \mathbf{v} + \varepsilon\varphi$, $\mathbb{Y} = (\mathbb{B}^{-1} + \varepsilon\Phi)^{-1}$ and limits $\varepsilon \to 0\pm$ recover the equations for **v** and $\mathbb{B}$. Hence, a relative energy inequality itself represents a “dual” weak formulation.

Open question: Does energy inequality imply relative energy inequality?

Theorem (to appear soon)

*For any initial data $\mathbf{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)$ and $\mathbb{B}_0 \in L^1(\Omega)$ positive definite such that $\psi(\mathbb{B}_0) \in L^1(\Omega)$, there exists a global-in-time, three-dimensional weak solution $(\mathbf{v}, \mathbb{B})$ to the system, satisfying also the **relative** energy inequality and fulfilling the boundary condition $\mathbf{v}\,|_{\partial\Omega} = \mathbf{w}_D$.*

Corollary

There exists $\delta > 0$ such that for any weak solution $(\boldsymbol{u}, \mathbb{A})$ fulfilling the boundary condition $\boldsymbol{u}\,|_{\partial\Omega} = \boldsymbol{w}_D$ and the smallness condition

$$\sup_{(0,\infty)\times\Omega} \left(|\nabla \boldsymbol{u}| + |\mathbb{A} - \mathbb{I}| + |\nabla \mathbb{A}|\right) \leq \delta,$$

there exists $\varepsilon > 0$, such that

$$L(t) \leq e^{\varepsilon(t_0 - t)} L(t_0), \qquad t \geq t_0, \quad \text{where} \quad L = \int_\Omega \left(\frac{1}{2}|\mathbf{v} - \boldsymbol{u}|^2 + \psi(\mathbb{A}^{-1}\mathbb{B})\right).$$