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# Boundary regularity for systems with symmetric gradients

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MPDE, September 2024
Prague

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# Overview

For $u : \mathbb{R}^n \to \mathbb{R}^n$, we study the system of PDEs

$$\begin{aligned} -\operatorname{div}(A(\varepsilon u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The system depends on the *symmetric gradient* $\varepsilon u := \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$.

The nonlinear map $A$ is given by, e.g.,

$$A(\varepsilon u) := \big(\delta + |\varepsilon u|\big)^{p-2} \varepsilon u,$$

where $p \in (1, \infty)$ and $\delta \geq 0$.

**Application:** Elasticity theory, model problem for the $p$-Stokes system

**Goal:** Regularity of the solution $u$ near the boundary $\partial\Omega$.

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# Known results: $p$-Laplace system

$u : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

If $f$ and $\Omega$ are regular enough, then

$$u \in C^{1,\alpha}(\overline{\Omega}) \qquad \text{and} \qquad V(\nabla u) := |\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}(\Omega)$$

was shown by

- *Uhlenbeck; 1977*: $2 \leq p < \infty$,
- *Acerbi, Fusco; 1989*: $1 < p < 2$,
- *DiBenedetto, Chen; 1989*: Up to the boundary,
- *Diening, Stroffolini, Verde; 2009*: For Orlicz-growth.

*Balci, Cianchi, Diening, Maz'ya; 2022*: $|\nabla u|^{p-2}\nabla u \in W^{1,2}(\Omega)$ for $p > 1.1715...$

# Known results: symmetric $p$-Laplace system

$$
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-\operatorname{div}(|\varepsilon u|^{p-2}\varepsilon u) &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

**Known results:**

- For $n = 2$: $u \in C^{1,\alpha}(\Omega)$, even for the $p$-Stokes-system:
  *Kaplický, Málek, Stará; 1999*, *Diening, Kaplický, Schwarzacher; 2014*
- $V(\varepsilon u) := |\varepsilon u|^{(p-2)/2}\varepsilon u \in W^{1,2}(\Omega)$:

  For $1 < p \leq 2$ near a flat boundary: *Seregin, Shilkin; 1997*
  Extended to $\partial\Omega \in C^{2,1}$ and parabolic systems: *Berselli, Růžička; 2017-2022*

**Open questions:**

- $u \in C^{1,\alpha}(\Omega)$ (for $n \geq 3$)?
- $|\varepsilon u|^{p-2}\varepsilon u \in W^{1,2}(\Omega)$ (for $p \neq 2$)?.

# Main theorem for $p$-growth

**Theorem (Behn, Diening '24)**

*Let $\Omega$ be a bounded $C^{2,1}$-domain, $1 < p < \infty$ and $f \in W_0^{1,p'}(\Omega)$. Then the system*

$$\begin{aligned} -\operatorname{div}(|\varepsilon u|^{p-2}\varepsilon u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

*has a unique weak solution $u \in W^{1,p}(\Omega)$ fulfilling $V(\varepsilon u) := |\varepsilon u|^{\frac{p-2}{2}}\varepsilon u \in W^{1,2}(\Omega)$ and*

$$\|V(\varepsilon u)\|^2_{W^{1,2}(\Omega)} \lesssim \|f\|^{p'}_{W^{1,p'}(\Omega)}.$$

- $p \in (1,2)$: $u \in W^{2,\frac{np}{n+p-2}}(\Omega)$
- $p \in (2,\infty)$: $u \in W^{1+\frac{2}{p},p}(\Omega)$

# Main theorem for Orlicz growth

We replace $\mathcal{J}_p(u) := \frac{1}{p}\int_\Omega |\varepsilon u|^p\,dx$ by energies with more general $\varphi$-growth

$$\mathcal{J}_\varphi(u) := \int_\Omega \varphi(|\varepsilon u|)\,dx.$$

**Theorem (Behn, Diening '24)**

*Let $\Omega$ be a bounded $C^{2,1}$-domain, $\varphi$ a uniformly convex N-function and $f \in W_0^{1,\varphi^*}(\Omega)$. Then the system*

$$\begin{aligned} -\operatorname{div}(A(\varepsilon u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

*has a unique weak solution $u \in W^{1,\varphi}(\Omega)$ fulfilling $V(\varepsilon u) \in W^{1,2}(\Omega)$ and*

$$\|V(\varepsilon u)\|^2_{W^{1,2}(\Omega)} \lesssim \int_\Omega \varphi^*(|f|) + \varphi^*(|\nabla f|)\,dx.$$

$$A(Q) := \varphi'(|Q|)\frac{Q}{|Q|}, \qquad V(Q) := \sqrt{\varphi'(|Q|)|Q|}\,\frac{Q}{|Q|}$$

# Korn inequality

Clearly, $|\varepsilon u(x)| \leq |\nabla u(x)|$. However, $|\nabla u(x)| \nleq c|\varepsilon u(x)|$.

**Theorem (Korn's inequality)**

*Let $p \in (1, \infty)$. Then there exists $c > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq c\|\varepsilon u\|_{L^p(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

For $p = 1$ and $p = \infty$ this inequality fails to hold (Ornstein, 1962).

Thus (informally):

*There are no problems coming from the function spaces involved!*

Instead, the difficulty lies in the higher coupling of the system.

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# Coupling of the equation

For $p = 2$:

- $-\Delta u = f \iff -\Delta u_i = f_i \quad \forall\, 1 \le i \le n$ (*system decouples*)
- $-\Delta^{\mathrm{sym}} u := -\operatorname{div}(\varepsilon u) = -\frac{1}{2}\Delta u - \frac{1}{2}\nabla \operatorname{div} u$ (*system is highly coupled*)

For $-\Delta u = f \in L^2$ we can show $u \in W^{2,2}$ near the boundary by

1. *For tangential directions $\alpha$:* Testing with $\partial_\alpha \partial_\alpha u$. This gives $\partial_\alpha \nabla u \in L^2$.

   **Does not work for the normal direction $\partial_n \nabla u$ !**

2. Using the PDE: $\partial_n \partial_n u = f - \sum_\alpha \partial_\alpha \partial_\alpha u \in L^2$.

**Main difficulty**: The high coupling and nonlinear nature of the system.

To overcome this, we rely on the algebraic identity $\partial_{ij} u_k = \partial_i \varepsilon_{jk} u + \partial_j \varepsilon_{ik} u - \partial_k \varepsilon_{ij} u$.

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# Summary

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$$\|V(\varepsilon u)\|^2_{W^{1,2}(\Omega)} \lesssim \int_\Omega \varphi^*(|f|) + \varphi^*(|\nabla f|)\, dx.$$

For $p$-growth: $p \le 2 : u \in W^{2,\frac{np}{n+p-2}}$, $p \ge 2 : u \in W^{1+\frac{2}{p},p}$.

**Thank you for your attention!**