Linearized elastodynamics

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Modelling, partial differential equations analysis and computational mathematics in material sciences

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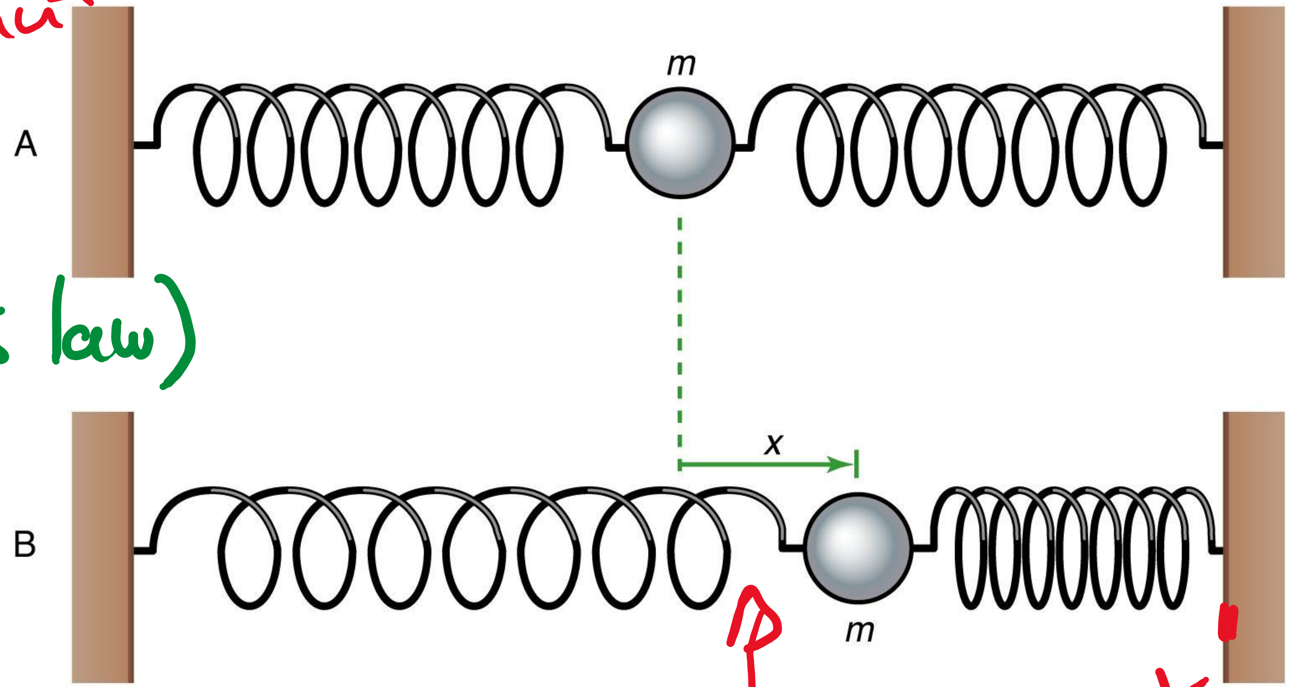
Harmonic oscillator

The force is a linear function of the displacement

"elastic constant"

$$f = -kx$$

(Hooke's law)



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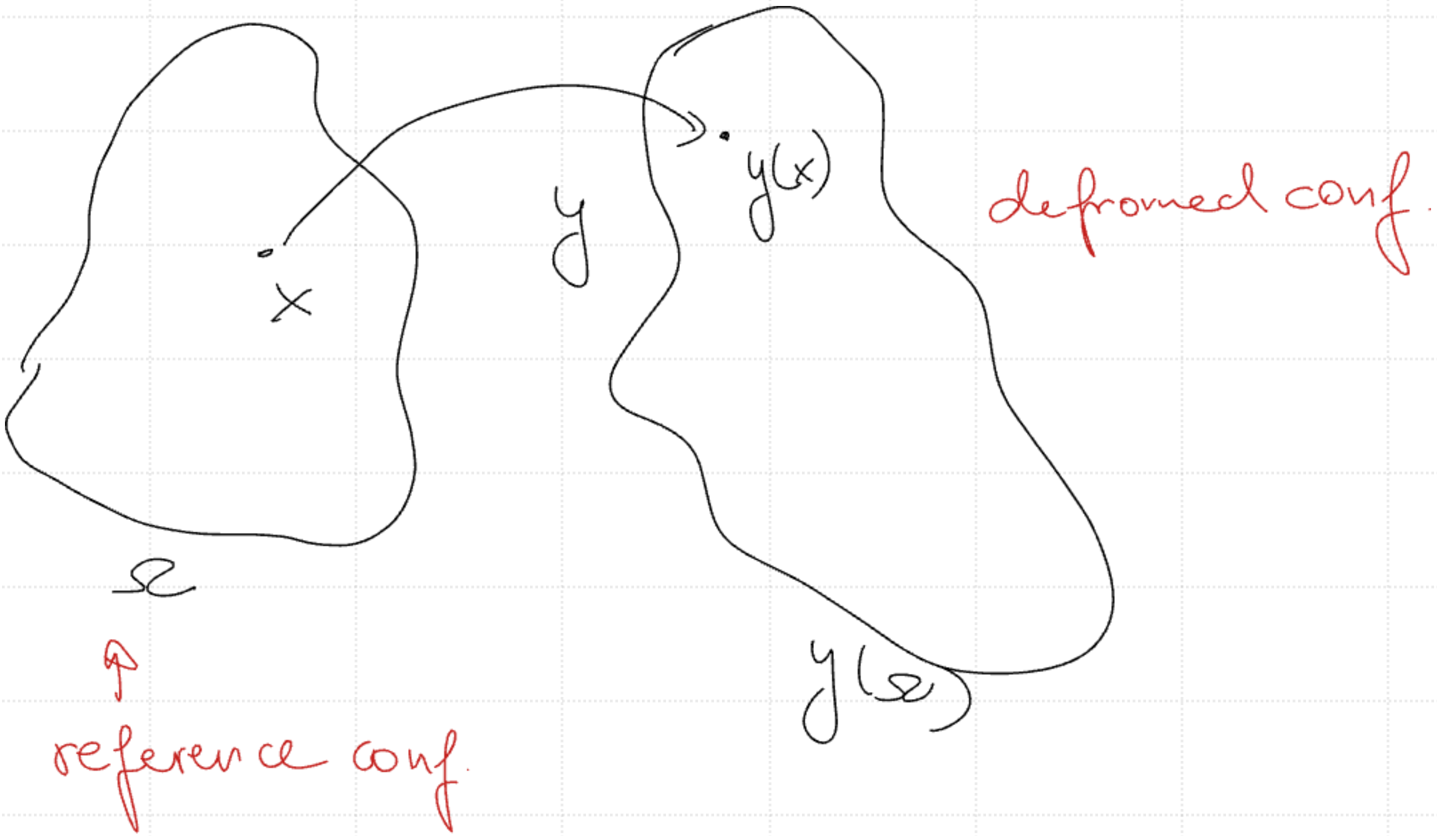
"displacement"

Therefore Newton's law can be written as

$$m\ddot{x} = -kx$$

mass

… in continuum mechanics



$$y(x) = x + \underbrace{u(x)}_{\text{displacement}}$$

… in continuum mechanics

The model in which the force is proportional to the stain is linearized elasticity

$$\rho \ddot{u} = f + \operatorname{div}\Big(\mathbb{C}\,\underbrace{\Big(\frac{\nabla u + \nabla u^T}{2}\Big)}_{\varepsilon \,\dots\, \text{strain}}\Big)$$

where

f … ext force

$\mathbb{C}$ … tensor of el. constants

The underlying energy:

$$W(\varepsilon) = \varepsilon \cdot (\mathbb{C}\,\varepsilon)$$

… in continuum mechanics

The model in which the force is proportional to the stain is linearized elasticity

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Only valid for small strains….

... large strain vs small strain

The model of linearized elasticity does not satisfy the physical principle of frame-indifference

$$W(QF) = W(F)$$

with $Q \in SO(3)$

$$F = \nabla y$$

and impossibility of infinite compression/orientation reversal

$$W(F) = +\infty \text{ if } \det F < 0$$

$$\& \quad W(F) \to \infty \text{ if } \det F \to 0^+$$

It only holds in the approximation when the displacement is very small so that

$$y(x) = x + \delta u(x) \qquad F^T F = I + \delta(\nabla u + \nabla u^T) + \cancel{\delta^2 \nabla u^T \nabla u}$$

$$\sigma = \partial_F W(F) = \partial_F W(I) + \delta \underbrace{\partial_F^2 W(I)}_{\mathbb{C}} \nabla u + \cancel{o(\delta^2)}$$

Rigorous derivation – energies

Assumption

$W(F) \geq \mathrm{dist}^2(F, SO(3))$, $W(I) = 0$, $\partial_F W(I) = 0$

Set $\quad I_\delta(u) = \frac{1}{\delta^2} \int_\Omega W(I + \delta \nabla u)\,dx$

Theorem [Dal Maso, Negri, Percivale]

- If $\quad I_\delta(u_\delta) \leq C \overset{\text{subs.}}{\Rightarrow} \exists\, u \in W^{1,2}$ s.t.
  $u_\delta \rightharpoonup u$ in $W^{1,2}$
- $I_\delta \xrightarrow{\Gamma} \frac{1}{2} \int_\Omega \varepsilon\, (\mathbb{C})\, \varepsilon \quad$ with $\varepsilon = \frac{(\nabla u + \nabla u^T)}{2}$

[Dal Maso, Negri, Percivale, '02]

Rigorous derivation – energies

What this tells us is that "minimizers converge to minimizers" $u_\delta \in \underset{\varphi \in W^{1,2}}{\arg\min}\, I_\delta(\varphi)$ $u_\delta \longrightarrow u$ so that

$$u \in \arg\min \int_\Omega \frac{1}{2} \varepsilon_u (\mathbb{C}\, \varepsilon_u)$$

It does not tell us

if u_δ solve

$$DI_\delta(u_\delta) = 0$$

$$\boxed{\text{then } u_\delta \to u \text{ \& } u \text{ solves " } \operatorname{div}\left(\mathbb{C}\left(\frac{\nabla u + \nabla u^T}{2}\right)\right) = 0 \text{ "}}$$

[Dal Maso, Negri, Percivale, '02], [Ball, '85]

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In fact, the Euler- Lagrange equation may be unrelated to the variational problem

[Dal Maso, Negri, Percivale, '02], [Ball, '85]

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It does not tell us

$$\text{if } u_\delta \text{ solve } DI_\delta(u_\delta) = 0$$

$$\boxed{\text{then } u_\delta \to u \ \&\ u \text{ solves } \text{“} \operatorname{div}\left(\mathbb{C}\left(\frac{\nabla u + \nabla u^T}{2}\right)\right) = 0 \text{”}}$$

An even if it were, equi-coerciviy is missing

[Dal Maso, Negri, Percivale, '02], [Ball, '85]

Fix I – obtaining the E.-L. eq.

Change the elastic elastic energy a bit

$$\tilde{I}_\delta(v_\delta) = \frac{1}{\delta^2}\int_\Omega W(I+\delta Dv_\delta)\,dx + \frac{1}{\delta^{p\alpha}}\int_\Omega |D^2 v_\delta|^p$$

with $p>3$, $\alpha\in(0,1)$

<u>Theorem</u> [Friedrich, Kružík]

We have that $I_\delta(v_\delta) \xrightarrow{\Gamma} \frac{1}{2}\int_\Omega \varepsilon(\mathbb{C}\varepsilon)\,dx$

In addition

$$\|Dy_\delta - I\|_{\underbrace{L^\infty(\Omega)}} \leq \delta^\alpha$$

Notice the L^∞!

[Healey, Krömer, 08],
[Friedrich, Kružík, '17]
[Friesecke, James, Müller, '02]

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In addition $\|Dy_\delta - I\|_{\underbrace{L^\infty(\Omega)}} \leq \delta^\alpha$

Notice the L^∞!

From now on, we will always use this modification

[Healey, Krömer, 08], [Friedrich, Kružík, '17] [Friesecke, James, Müller, '02]

Fix II – equicoercivity

Still we face the problem of coercivity

for solutions $\{u_\delta\}_{\delta>0}$ of $DI_\delta(u_\delta) = 0$

we don't know $\underbrace{I_\delta(u_\delta) \le C}$

needed for a-priori est.

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We actually want to consider the dynamic problem...

Dynamic model – variational approx.

Our problem can "informally" be expressed as

$$\rho \ddot{u}_\delta = \cancel{f_\delta} + D\mathcal{I}_\delta(u_\delta)$$

This can be approximated as

$$\rho \frac{\dot{u}(t) - \dot{u}(t-h)}{h} = D\mathcal{I}_\delta(u_\delta(t)) \quad \text{for } h>0$$

"data $= f(t)$"

discretize again for h fixed

[B., Kampschulte,Schwarzacher, '23]

Dynamic model – variational approx.

Our problem can "informally" be expressed as

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This actually has only measure-valued solution, we should add dissipation

This can be approximated as

$$\rho \frac{\dot{v}(t) - \dot{v}(t-h)}{h} = D\mathcal{I}_\delta(v_\delta(t)) \quad \text{for } h>0$$

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discretize again for h fixed, so

$0 \le \tau \le 2J \ldots \le NJ = h$ &

$$\rho \frac{u^k - u^{k-1}}{Jh} = D I_\delta (u_k) + f(Jk)$$

Euler-Lagrange equation

$$\min_u \left[I_\delta(u) - \frac{\rho}{h} J \left\| \frac{u - u^{k-1}}{J} \right\|^2 + \int_\Omega f u \right]$$

[B., Kampschulte,Schwarzacher, '23], [DeGiorgi, '93]

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$0 \le \tau \le 2\tau \ldots \le N\tau = h$ &

$$\rho \, \frac{u^k - u^{k-1}}{\tau h} = D\mathcal{I}_\delta(u_k) + f(\tau k)$$

Euler-Lagrange equation

$$\min_u \left[\mathcal{I}_\delta(u) - \frac{\rho}{h}\tau \left\| \frac{u - u^{k-1}}{\tau} \right\|^2 + \int_\Omega f u \right]$$

Discrete nonlinear problem

[B., Kampschulte,Schwarzacher, '23], [DeGiorgi, '93]

Dynamic model – weak solution

Thm [B. Kampschulte, Schwarzacher]

The solutions to the disc. nonlinear problem converge upon taking $\tau \to 0$ & then $h \to 0$ to the weak sol. of

1) $\rho \ddot{u}_\delta = D I_\delta(u_\delta)$ & additionally

2) $\frac{\rho}{2}\|\dot{u}_\delta(t)\| + I_\delta(u_\delta(t)) \leq \frac{\rho}{2}\|\dot{u}_0\| + I_\delta(u_0)$

[B., Kampschulte,Schwarzacher, '23], [Demoulini' 01]

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Technically, this is again cheating, because we should have included dissipation

[B., Kampschulte,Schwarzacher, '23], [Demoulini' 01]

Dynamic model – weak solution

Thm [B. Kampschulte, Schwarzacher]

The solutions to the disc. nonlinear problem converge upon taking $J \to 0$ & then $h \to 0$ to the weak sol. of

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2) $\frac{\rho}{2} \|\dot{u}_\delta(t)\| + I_\delta(u_\delta(t)) \leq \frac{\rho}{2} \|\dot{u}_0\| + I_\delta(u_0)$

Notice that we now have the energies equibounded

[B., Kampschulte, Schwarzacher, '23], [Demoulini' 01]

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The solutions to the disc. nonlinear problem converge upon taking $\tau \to 0$ & then $h \to 0$ to the weak sol. of

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Weak solutions of (visco)elastodynamics can also be approached by other methods, but...

[B., Kampschulte,Schwarzacher, '23], [Demoulini' 01]

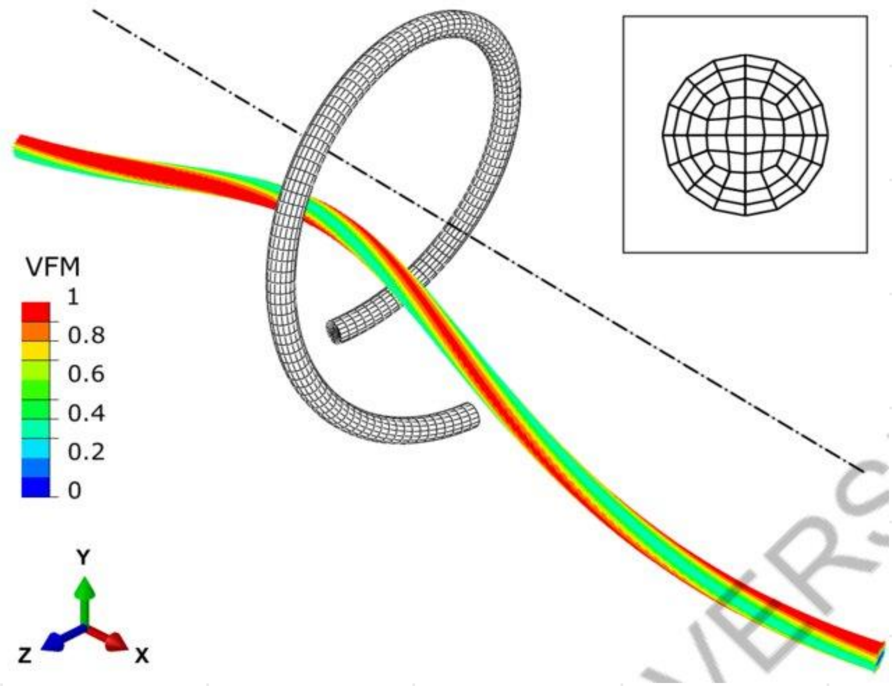
Commutative diagram

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Discrete nonlinear prb.} \arrow[r, "\substack{\sigma \to 0 \\ h \to 0}"] \arrow[d, "\substack{\Gamma\text{-limit} \\ \text{[Dal Maso, Negri, Percivale]}}"] & \text{Nonlinear elastodynamics} \arrow[d, "\text{equibounded}"] \\
\min\limits_{v} \int_\Omega \left[\mathcal{E}(\nabla \varepsilon) + f v \right] + \frac{\rho\sigma}{h} \left\| \frac{v - v^{k-1}}{\sigma} \right\|^2 \arrow[r, "\substack{\sigma \to 0 \\ h \to 0 \\ \text{(here easy, linear)}}"'] & \rho \ddot{u} = \operatorname{div}(\nabla \varepsilon)
\end{tikzcd}
$$

[B., Kampschulte, Kružík, '24?]

Why study limits to the linearized setting?

- Problably the most used model in solid mechanics in engineering
- Can be coupled to inner variables, dissipation phenomena etc.
- Easy to verify, *"only"* 1 set of paramenters
- The passage to the linearized setting in a sense justifies the non-linear setting



[Frost et al, '20]

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Determination of All 21 Independent Elastic Coefficients of Generally Anisotropic Solids by Resonant Ultrasound Spectroscopy: Benchmark Examples

P. Sedlák · H. Seiner · J. Zídek · M. Janovská · M. Landa

Received: 24 October 2013 / Accepted: 5 February 2014 / Published online: 11 March 2014

Abstract We present an experimental methodology for determination of all 21 elastic constants of materials with general (triclinic) anisotropy. This methodology is based on contactless resonant ultrasound spectroscopy complemented by pulse-echo measurements and enables full characterization of elastic anisotropy of such materials from measurements on a single small specimen of a parallelogram shape. The methodology is applied to two benchmark

reduces this number to 13 (monoclinic materials). The real triclinic symmetry class applies e.g. for certain types of minerals (talc [2] or albite [3]), superconducting iron-arsenides [4], or advanced perovskites [5, 6], etc. There have not been any successful attempts to determine full elastic tensors of such truly triclinic materials reported in the literature yet, although the well established ultrasonic methods (such as pulse-echo method [7], through-transmission

Thank you for your attention!

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