INITIAL VALUE PROBLEMS BY SPACE-TIME CONVEX OPTIMIZATION

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I will start with the simplest example: the (quadratic) porous medium equation. Then, I will discuss the challenging case of the Einstein equation in vacuum with a cosmological constant revisited as a matrix valued version of the isothermal Euler equations.

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(QPME) $\partial_t u = \Delta u^2/2, \quad u = u(t,x) \geq 0, \; t \geq 0, \; x \in \mathbb{T}^d,$

which is nothing but the macroscopic limit of the properly rescaled (deterministic) system of particles:

$$\frac{dX_k}{dt} = \epsilon^{-1} \sum_{j=1,N} (X_k - X_j) \exp\left(-\frac{|X_k - X_j|^2}{\epsilon}\right),$$

$$u(t,x) \sim \frac{1}{N} \sum_{j=1,N} \delta(x - X_j(t)), \quad 1/N << \epsilon^d << 1.$$

see for instance P.-L. Lions, S. Mas-Gallic 2001.

We start with the strange minimization problem:

$$\inf \int_Q u^2(t,x)dxdt, \quad Q = [0,T] \times \mathbb{T}^d,$$

among all WEAK solutions $u \in L^2(Q)$ ot the QPME

$$\partial_t u = \Delta u^2/2, \quad u = u(t,x) \in \mathbb{R}, \quad t \geq 0, \quad x \in \mathbb{T}^d,$$

with a smooth given initial condition $u_0 > 0$.

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$$I(u_0) = \inf_u \ \sup_\phi \int_Q \left(u^2 - 2\partial_t \phi u - \Delta\phi \ u^2 + 2u_0 \partial_t \phi \right),$$

where the only constraints are:
i) for test function ϕ to be smooth and vanish at $t = T$;
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This problem admits an interesting concave relaxation:

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$$J(u_0) = \sup_{\phi} \inf_{u} \int_Q \left(u^2 - 2\partial_t\phi u - \Delta\phi\, u^2 + 2u_0\partial_t\phi\right) =$$

$$\sup_{\phi} \int_Q \left(-\frac{(\partial_t\phi)^2}{1-\Delta\phi} + 2u_0\partial_t\phi\right), \quad \Delta\phi \leq 1, \quad \phi(T,\cdot) = 0.$$

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Setting $q = \partial_t\phi$, $\sigma = 1 - \Delta\phi$, we get: $\quad J(u_0) =$

$$\sup_{\sigma,q} \int_Q \left(-\frac{q^2}{\sigma} + 2u_0\, q\right), \quad \partial_t\sigma + \Delta q = 0, \quad \sigma(T,\cdot) = 1$$

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s.t.

$$\partial_t \sigma + \Delta q = 0, \quad \sigma(T,\cdot) = 1,$$

which is (at least for $d = 1$) the "ballistic" version of the formulation proposed by Huesmann and Trevisan for the martingale optimal transport problem:

"A Benamou-Brenier formulation of martingale optimal transport" Bernoulli 2019.

(See also Loeper, Ghoussoub-Kim and the recent work made in the Mokaplan team.)

The proof is based on a simple remark:

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Indeed, let us find a solution ϕ of the "adjoint" problem

$$\partial_t \phi = (1 - \Delta\phi)u, \quad \phi(T, \cdot) = 0,$$

i.e., for $\alpha = 1 - \Delta\phi$: $\partial_t \alpha + \Delta(\alpha u) = 0, \quad \alpha(T, \cdot) = 1.$

From Aronson-Bénilan, we deduce $\alpha(t, x) \geq (t/T)^\kappa$.

Proof. (Assuming u to be smooth) we have

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$$A'(t) \leq \kappa A(t)/t.$$

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So, $\log A(T) - \log A(t) \leq \kappa(\log T - \log t)$, and therefore

$A(t) \geq (t/T)^\kappa$ (since $A(T) = 1$). End of proof.

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(using $\partial_t \phi = (1 - \Delta \phi)u$) which shows that ϕ is optimal since $J(u_0) \geq j = \int_Q u^2 \geq I(u_0) \geq J(u_0)$.

II. Example of the isothermal Euler equations: Solve the initial value problem by minimizing

$$\int_{[0,T]\times\mathbb{T}^d} \exp(u)\exp(\frac{1}{2}Z\cdot M^{-1}\cdot Z) + \int_{\mathbb{T}^d} \sigma_0\rho_0 + w_0\cdot\rho_0 v_0$$

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among all fields $u = u(t,x) \in \mathbb{R},\ Z = Z(t,x) \in \mathbb{R}^d,$ $M = M(t,x) = M^T(t,x) \in \mathbb{R}^{d\times d}, \quad M \geq 0,$

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Proposition (YB 18): Smooth solutions with initial data ρ_0, v_0 can be recovered this way, for small $T > 0$.

III. Vacuum Einstein with cosmological constant

"Free fall" in a Lorentzian metric g is described by ("constant speed") geodesics $s \in \mathbb{R} \to x(s) \in \mathbb{R}^4$

i.e. critical points of $\displaystyle\int \frac{dx(s)}{ds} \cdot g(x(s)) \cdot \frac{dx(s)}{ds} ds$ i.e.

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$$\frac{dx^i(s)}{ds} = \xi^i(s), \quad \frac{d\xi^i(s)}{ds} = -\Gamma^i_{jk}(x(s))\xi^j(s)\xi^k(s),$$

$$2g_{im}\Gamma^m_{jk} + \partial_i g_{jk} - \partial_j g_{ik} - \partial_k g_{ij} = 0, \quad i,j,k,m \in \{0,1,2,3\},$$

Γ^m_{jk} being just the Christoffel symbols of g.

Our trick: write everything on the "tangent bundle"

$$(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4, \;\; V^j_k(x, \xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma \quad (j, k, \gamma \in \{0, 1, 2, 3\})$$

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so that the Riemann and the Ricci curvatures just read

$$R^n_{jkm}(x)\xi^m = \Big((\partial_{x^k} + V^\gamma_k \partial_{\xi^\gamma})V^n_j - (\partial_{x^j} + V^\gamma_j \partial_{\xi^\gamma})V^n_k\Big)(x,\xi)$$

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The Einstein equations in vacuum with cosmological constant just mean for the Lorentzian metric g over $\mathbb{R}^4$ that its Ricci curvature is proportional to g.

Similarly, we may encode the link between Γ and g:

$$\partial_{x^j} g_{kq} = \Gamma^m_{jk} g_{mq} + \Gamma^m_{jq} g_{mk}, \quad \Gamma^i_{jk} = \Gamma^i_{kj},$$

which indeed is equivalent to $\quad 2g_{mi}\Gamma^i_{jk} + \partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mj} = 0,$

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by $\partial_{x^j}\rho + \partial_{\xi^m}(\rho V^m_j) = 0,$

with

$$\rho(x,\xi) = \exp\left(\frac{g_{ij}(x)\xi^i\xi^j + \log|\mathrm{det} g(x)|}{2}\right), \quad V^j_k(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma$$

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i.e. in compact *non geometric* notation:

$$\nabla_x \rho + \nabla_\xi \cdot (\rho V) = 0.$$

Proposition

The Einstein equations with cosmological constant Λ just describe special solutions, linear/log-quadratic in ξ

$$V(x,\xi) = -\Gamma(x)\cdot\xi,\ \ \rho(x,\xi) = \sqrt{|\mathrm{det} g(x)|}\,\exp(\frac{\xi\cdot g(x)\cdot\xi}{2})$$

of the LIFTED PDE system set on $(x,\xi)\in\mathbb{R}^4\times\mathbb{R}^4$:

$$\nabla_x\rho + \nabla_\xi\cdot(\rho V) = 0,\quad V\in\mathbb{R}^{4\times 4},\quad \rho\in\mathbb{R}_+,$$

$$\nabla_x\cdot(\rho[V]) + \nabla_\xi\cdot(\rho[V]V) = \Lambda\nabla_\xi\rho,$$

where $[M] = M - \mathbb{I}_4\,\mathrm{trace}(M)$ for $M\in\mathbb{R}^{4\times 4}$, ∇_x, ∇_ξ being just plain Euclidean gradients and $\nabla_x\cdot M$, $\nabla_\xi\cdot M$ standing for $\partial_{x^j}M^j_k$, $\partial_{\xi^j}M^j_k$.

There is a striking similarity between the LIFTED EINSTEIN equations with cosmological constant Λ

$$\nabla_x \rho + \nabla_\xi \cdot (\rho V) = 0, \quad V \in \mathbb{R}^{4\times 4}, \quad \rho \in \mathbb{R}_+,$$

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and the EULER equations of isothermal compressible fluids with constant sound speed c

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \; \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = -c^2 \nabla \rho.$$

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$$(t,x) \in \mathbb{R}^{1+d} \to (x,\xi) \in \mathbb{R}^{4+4}, \ \partial_t \to \nabla_x, \ \nabla \to \nabla_\xi,$$
$$v \in \mathbb{R}^d \to V \in \mathbb{R}^{4\times 4}, \quad \rho \in \mathbb{R}_+ \to \rho \in \mathbb{R}_+, \quad c^2 \to -\Lambda.$$

"LISEZ EULER, IL EST NOTRE MAITRE A TOUS!" (Laplace)

For the isothermal Euler equations, we saw how to solve the initial value problem by minimizing

$$\int_{[0,T]\times\mathbb{T}^d} \exp(u)\exp\left(\frac{1}{2}Z\cdot M^{-1}\cdot Z\right) + \int_{\mathbb{T}^d} \sigma_0\rho_0 + w_0\cdot\rho_0 v_0$$

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For Einstein's equations, a very similar formula can be worked out!

Děkuji za pozornost!