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Flow around an obstacle:

Various approaches to calculate pointwise traction

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Point-wise traction

- Flow force inducing local deformation of bodies
- Net forces (lift, drag) insufficient
- Fluid-Structure Interaction (FSI)
- Estimate of initial deformation without FSI
- No benchmarks yet

Turek Benchmark

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- Cannonical computational benchmark for Navier-Stokes equation
- Flow around cylinder
- Provides referential values for lift and drag only
- **Aim:** Compare different approaches for computing point-wise traction

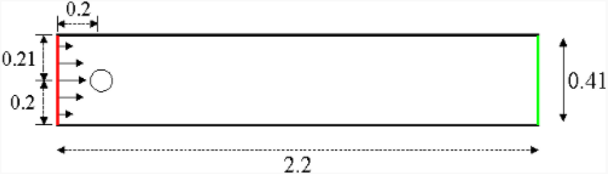

Figure: Turek, Schaefer; Benchmark computations of laminar flow around cylinder; in Flow Simulation with High-Performance Computers II, Notes on Numerical Fluid Mechanics 52, 547-566, Vieweg 1996

Equations and traction

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- Steady incompressible Navier-Stokes equations

$$
\begin{aligned}
\rho \mathbf{v} \cdot \nabla \mathbf{v} &= \operatorname{div} \mathbb{T} \quad \text{in } \Omega, \\
\operatorname{div} \mathbf{v} &= 0 \quad \text{in } \Omega, \\
\mathbb{T} &= -p\mathbb{I} + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T), \\
\mathbf{v} &= \mathbf{v}_D^i \quad \text{on } \Gamma_i \subset \partial\Omega, \\
\mathbb{T}\mathbf{n} &= 0 \quad \text{on } \partial\Omega \backslash \cup \Gamma_i.
\end{aligned}
$$

- Traction

$$\mathbf{t} := \mathbb{T}\mathbf{n}.$$

Traction computation

- Direct approach
  - Compute solution $\mathbf{v}, p$
  - Evaluate traction from definition $\mathbf{t} := \mathbb{T}(\nabla \mathbf{v}, p)\mathbf{n}$
- Weak / Dual / Poincaré-Steklov operator approach
  - Compute solution $\mathbf{v}, p$
  - Use equation again!

$$\int_\Omega \rho(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \phi \, dx = -\int_\Omega \mathbb{T} \cdot \nabla \phi \, dx + \int_{\partial\Omega} (\mathbb{T}\mathbf{n}) \cdot \phi \, dS.$$

  - Make new unknown $\mathbf{t}^{\mathrm{PS}}$ to solve for

$$\int_{\partial\Omega} \mathbf{t}^{\mathrm{PS}} \cdot \phi \, dS = \int_\Omega \mathbb{T} \cdot \nabla \phi \, dx + \int_\Omega \rho(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \phi \, dx$$

Analysis result

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Standard estimate for Stokes equation

$$\|p - p_h\|_{L^2(\Omega)} + \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2(\Omega)} \leq Ch\|\nabla^2 \mathbf{v}\|_{L^2(\Omega)} + Ch\|\nabla p\|_{L^2(\Omega)}$$

Scaling argument on the boundary (direct computation)

$$\|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2(\partial\Omega)} \leq C\|\mathbf{v} - \mathbf{v}_h\|_{H^{3/2}(\Omega)} \leq Ch^{\frac{1}{2}}\|\nabla^2 \mathbf{v}\|_{L^2(\Omega)}$$

We hope: Poincaré-Steklov approach retains former convergence rate

$$\|\mathbf{t}^{\mathsf{PS}} - \mathbf{t}_h^{\mathsf{PS}}\|_{L^2(\partial\Omega)} \leq Ch\|\nabla^2 \mathbf{v}\|_{L^2(\Omega)}$$

Conjecture

This also holds for the Navier-Stokes equation.

Implementation

- Firedrake – customizable finite element library
- Taylor-Hood pair
- Monolithic approach – Newton solver and sparse LU factorization
- Reference obtained on mesh with 7M DoFs

Implementation: Traction Computation

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- Poincaré–Steklov (PS): using CG1 elements solve for

$$\int_{\partial\Omega} \mathbf{t}^{\mathsf{PS}} \cdot \phi \, dS = \int_{\Omega} \mathbb{T} \cdot \nabla\phi \, dx + \int_{\Omega} \rho(\mathbf{v} \cdot \nabla\mathbf{v}) \cdot \phi \, dx$$

- L^2–projection (proj): using CG1 elements solve for

$$\int_{\partial\Omega} \mathbf{t}^{\mathsf{proj}} \cdot \phi \, dS = \int_{\Omega} \mathbb{T}\mathbf{n} \cdot \phi \, dx$$

- Direct (dir): interpolate values in nodes

$$\mathbf{t}^{\mathsf{dir}} = \mathbb{T}\mathbf{n}|_{\partial\Omega}$$

Convergence plots: Turek benchmark

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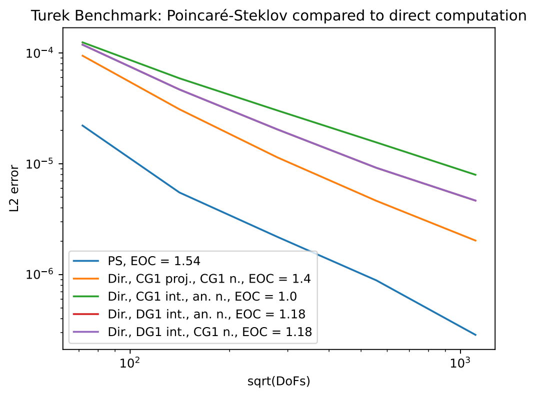
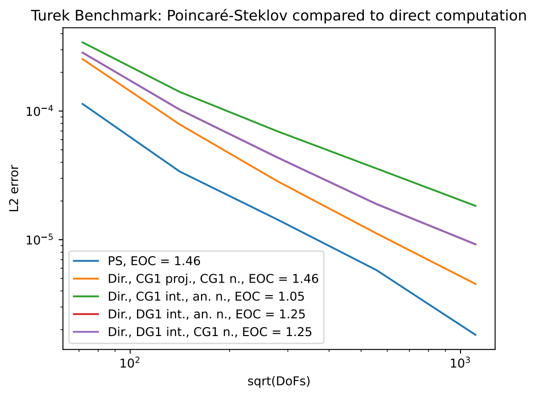
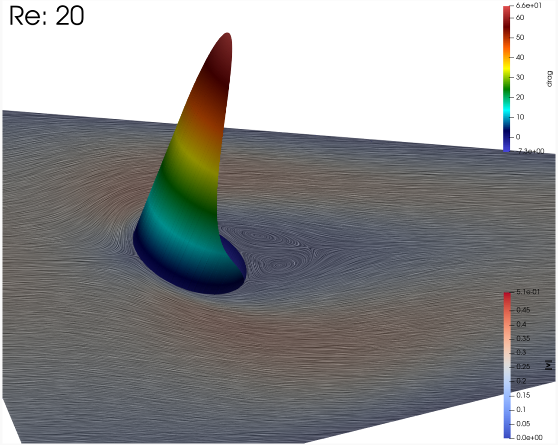


a) Stokes equations

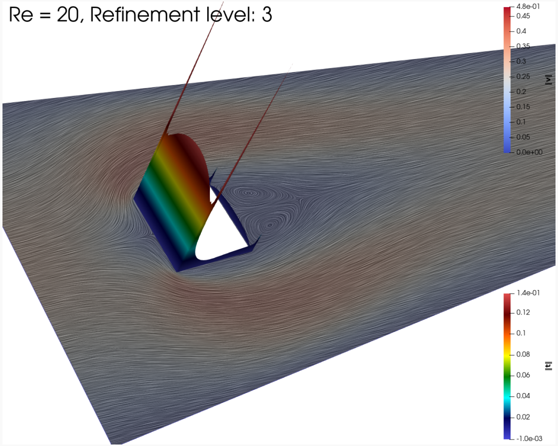


b) Navier-Stokes equations

Point-wise drag for Turek benchmark



Turek square benchmark



Convergence plots: Turek square bench.

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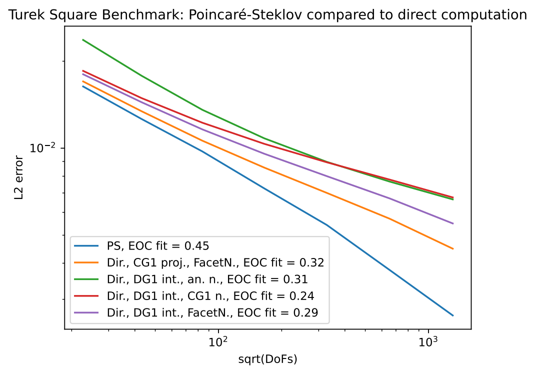
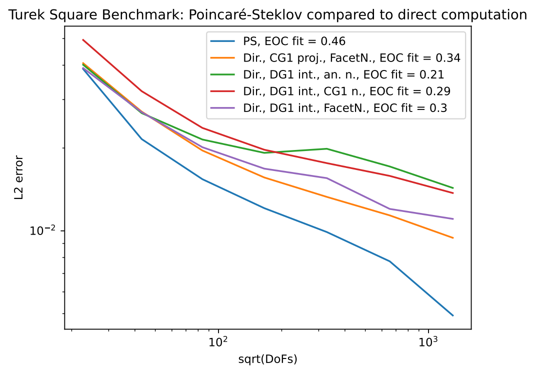


a) Stokes equations



b) Navier–Stokes equations

References and acknowledgement

1) Turek, Schaefer; *Benchmark computations of laminar flow around cylinder; in Flow Simulation with High-Performance Computers II*, Notes on Numerical Fluid Mechanics 52, 547-566, Vieweg 1996

2) David A. Ham et al. *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

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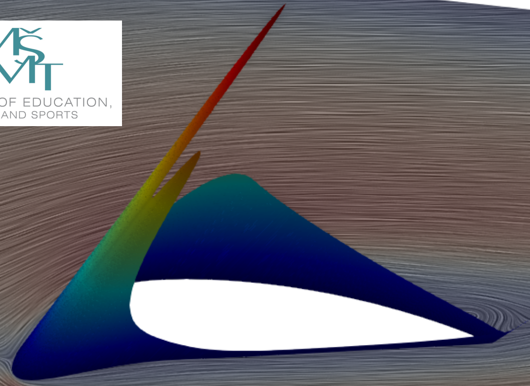

Questions?

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