Conditional regularity for the Navier-Stokes-Fourier system with applications

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Modelling, PDE analysis and computational mathematics in materials science, Praha
23 - 27 September, 2024

Czech Academy of Sciences

Nečas Center for Mathematical Modeling

GAČR CZECH SCIENCE FOUNDATION

Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance (Newton' s second law)

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \nabla_x G$$

Internal energy balance (First law of thermodynamics)

$$\partial_t \varrho e(\varrho, \vartheta) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q}(\nabla_x \vartheta) = -\kappa \nabla_x \vartheta, \ \kappa > 0$$

Thermodynamics

Gibbs' law, Second law of thermodynamics

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance equation (Second law of thermodynamics)

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Thermodynamic stability

$$(\varrho, S, \mathbf{m}) \mapsto \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S)\right] \text{ strictly convex, } S = \varrho s,\ \mathbf{m} = \varrho\mathbf{u}$$

Boyle-Mariotte equation of state

$$p(\varrho,\vartheta) = \varrho\vartheta,\ e(\varrho,\vartheta) = c_v\vartheta,\ c_v > 0,\ s(\varrho,\vartheta) = c_v\log\vartheta - \log\varrho$$

Data

Physical space

$$\Omega \subset R^d, \ d = 2, 3 \text{ (bounded) domain}$$

Inhomogeneous (no–slip) conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \ \ \mathbf{u}_B \cdot \mathbf{n} = 0 \text{ impermeable boundary}$$

Boundary temperature vs. heat flux

$$\Omega = \Gamma_D \cup \Gamma_N, \ \vartheta|_{\partial\Gamma_D} = \vartheta_B, \ \mathbf{q} \cdot \mathbf{n}|_{\Gamma_N} = 0$$

Initial state of the system

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \varrho_0 > 0, \vartheta_0 > 0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

$$+ \text{ compatibility conditions}$$

Initial/boundary value problem

Existence of local–in–time strong solutions

- **Valli, Valli–Zajaczkowski [1986], Kagei–Kawashima [2006]**

$$\varrho_0 \in W^{k,2}(\Omega),\ \vartheta_0 \in W^{k,2}(\Omega),\ \mathbf{u}_0 \in W^{k,2}(\Omega; R^d),\ k \geq 3$$

- **Cho–Kim [2006]**

$$\varrho_0 \in W^{1,p}(\Omega),\ \vartheta_0 \in W^{2,2}(\Omega),\ \mathbf{u}_0 \in W^{2,2}(\Omega; R^d),\ 3 < p \leq 6$$

$$\mathbf{u}_B = 0,\ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

- **Kotschote [2015]**

$$\varrho_0 \in W^{1,p}(\Omega),\ \vartheta_0 \in W^{2-\frac{1}{p},p}(\Omega),\ \mathbf{u}_0 \in W^{2-\frac{1}{p},p}(\Omega; R^d),\ p > 3$$

Conditional regularity results, I.

John F. Nash
[1928-2015]

Nash's conjecture: *Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density.*

- **EF, Wen, Zhu [2022] [Cho–Kim regularity class]**

$$\mathbf{u}_B = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

$$\sup_{t\in[0,T)}\left(\sup_{\Omega}\varrho(t,\cdot) + \sup_{\Omega}\vartheta(t,\cdot)\right) < \infty \ \Rightarrow\ T_{\max} > T$$

- **Basarić, EF, Mizerová [2023] [Valli–Zajaczkowski regularity class]**

$$\mathbf{u}_B\cdot\mathbf{n} = 0,\ \vartheta|_{\partial Q} = \vartheta_B$$

$$\sup_{t\in[0,T)}\left(\sup_{\Omega}\varrho(t,\cdot) + \sup_{\Omega}\vartheta(t,\cdot) + \sup_{\Omega}|\mathbf{u}(t,\cdot)|\right) < \infty \ \Rightarrow\ T_{\max} > T$$

Conditional regularity results, II.

Abbatiello, Basarić, Chaudhuri, EF [2024]

$$\mathbf{u}_B \cdot \mathbf{n} = 0,\ \vartheta|_{\Gamma_D} = \vartheta_B,\ \mathbf{q} \cdot \mathbf{n}|_{\Gamma_N} = q_B$$

$L^p - L^q$ class of solutions

$$3 < q < \infty,\ \frac{2q}{2q-3} < p < \infty$$

$$\varrho_0 \in W^{1,q}(\Omega),\ \vartheta_0 \in B^{2(1-\frac{1}{p})}_{q,p}(\Omega),\ \mathbf{u}_0 \in B^{2(1-\frac{1}{p})}_{q,p}(\Omega; R^3)$$

Blow–up criterion
Either $T_{\rm max} = \infty$ or

$$\limsup_{t \to (T_{\rm max})-} \|(\varrho, \vartheta, \mathbf{u})(t, \cdot)\|_{C(\overline{\Omega}; R^5)} = \infty.$$

The blow-up time is the same for **Cho-Kim**, **Valli–Zajaczkowski**, and **Kotschote** class

Lax equivalence principle in numerical analysis

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Lax equivalence principle - nonlinear version

- **Stability** - uniform L^∞ bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to a generalized solution

measure–valued solution

- **Weak–strong uniqueness** - the measure valued solution coincides with the strong solution on its life span $[0, T)_{\max}$
- **Conditional regularity** - $T_{\max} = \infty$ as long as the solution remains bounded

$$\Longrightarrow$$

- Unconditional convergence of bounded consistent approximations

Numerical problems with uncertain data

Probability space

$$\{\Omega; \mathcal{B}, \mathbb{P}\},\ \Omega \text{ measurable space}$$

$$\mathcal{B}\ \sigma - \text{algebra of measurable sets},\ \mathbb{P} - \text{complete probability measure}$$

Random data

$$\omega \in \Omega \mapsto D \in X_D \text{ Borel measurable mapping}$$

Solutions as random variables

$$T_{\max} = T_{\max}[D] - \text{random variable}$$

$$D \mapsto (\varrho, \vartheta, \mathbf{u})[D] \text{ random variable}$$

Statistical solution

$$\text{strong sense: } \omega \in \Omega \mapsto (\varrho, \vartheta, \mathbf{u})(t, \cdot)[D],\ t \in [0, T_{\max})$$

$$\text{weak sense: } \mathcal{L}[(\varrho, \vartheta, \mathbf{u})(t, \cdot)[D]]$$

$\mathcal{L}$ - law (distribution) of $(\varrho, \vartheta, \mathbf{u})(t, \cdot)$ in $W^{3,2}(Q) \times W^{3,2}(Q) \times W^{3,2}(Q; R^d)$

Convergence of consistent approximations

Strong data convergence

$$D_n = [\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to D = [\varrho_D, \vartheta_D, \mathbf{u}_D] \text{ in } X_D$$

$$\mathbb{P} - \text{ a.s.}$$

Consistent approximation

$$(\varrho_n, \vartheta_n, \mathbf{u}_n) = (\varrho, \vartheta, \mathbf{u})_{h_n}[D_n] \text{ a sequence of consistent approximations}$$

Hypothesis of boundedness in probability

For any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left\{ \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]| > M \right\} < \varepsilon$$

Convergence of consistent approximations, I

1. Apply Skorokhod representation theorem to the sequence $(D_n, \varrho_n, \vartheta_n \mathbf{u}_n, \Lambda_n)_{n=1}^{\infty}$,

$$\Lambda_n = \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]|$$

2. New sequence of data $\widetilde{D}_n$ with the same law on the standard probability space,

$$\widetilde{D}_n \to \widetilde{D} \text{ in } X_d, \ \mathrm{d}y \text{ surely.}$$

$$\widetilde{\Lambda}_n = \sup_{(0,T)\times Q} \varrho_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} \vartheta_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[\widetilde{D}_n]| \to \widetilde{\Lambda}$$

$$\mathrm{d}y \text{ surely}$$

$$\varrho_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\varrho} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q)$$

$$\vartheta_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\vartheta} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q)$$

$$\mathbf{u}_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\mathbf{u}} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q; R^d)$$

$$\mathrm{d}y \ \text{ surely}$$

Convergence of consistent approximations, II

4. Show the limit is a measure–valued solution with the data $\widetilde{D}$ in the sense of Březina, EF, Novotný [2020], see also Chaudhuri [2022]
5. Apply the weak–strong uniqueness principle to conclude the $(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}})$ is the unique strong solution associated to the data $\widetilde{D}$,

$$(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}}) = (\varrho, \vartheta, \mathbf{u})[\widetilde{D}].$$

   Conclude there is no need of subsequence, $T_{\rm max}[\widetilde{D}] > T$, and convergence is strong for in L^q for any finite q.
6. Pass to the original space using Gyöngy–Krylov theorem

Conclusion – unconditional convergence of consistent approximations

$$T_{\rm max}[D] > T \text{ a.s.}$$

$$(\varrho, \vartheta, \mathbf{u})_{h_n}[D_n] \to (\varrho, \vartheta, \mathbf{u})[D]$$

in $L^q((0,T) \times Q; R^{d+2})$ for any $1 \leq q < \infty$

in probability