Elastic axisymmetric necking in a stretched circular membrane

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in collaboration with

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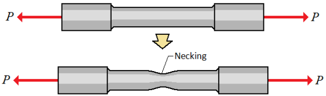
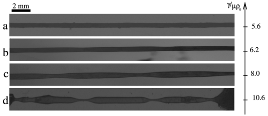
Funding: EPSRC, NSFC

In hard materials, necking is usually induced/accompanied by plastic deformations:

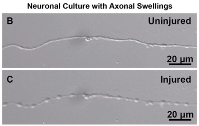


Elastic necking/bulging may also occur in soft materials subject to mechanical as well as non-mechanical fields (surface tension, electric field, etc).

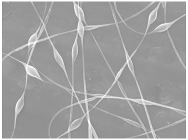
e.g. hydro-gels, nerve fibres, nanofibers during electrospinning, etc.



Mora *et al* (2010)



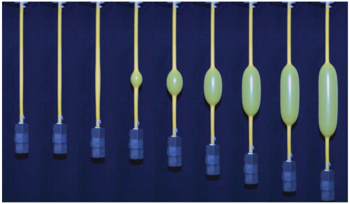
Hemphill *et al* (2015)



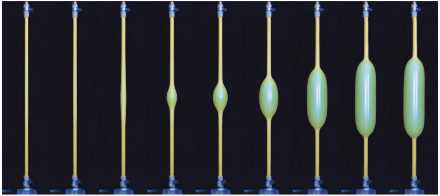
Fong *et al* (1999)

Fu, Jin & Goriely (JMPS, 2021).

Localized bulging in an inflated rubber tube:



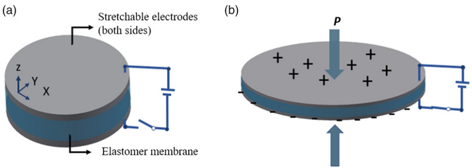
fixed axial force



fixed ends/length

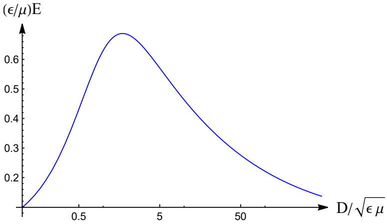
Wang *et al.* (JMPS, 2019)

Dielectric elastomer actuators (DEAs):



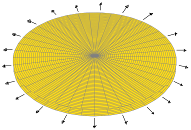
Wiranata et al (2021, Adv. Eng. Mater.)

Electric field E vs electric displacement D:

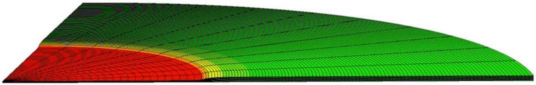


Based on the above curve, it is commonly believed that when E reaches the maximum, rapid but uniform thinning will take place, that leads to electric breakdown.

Current study: explore failure through axisymmetric necking



A typical Abaqus simulation



Outline

1. Governing equations and linear analysis
2. Weakly nonlinear analysis
3. Abaqus simulations: fully nonlinear regime
4. 1D reduced model
5. Summary

Based on three papers:

Purely mechanical case

Mi Wang, Lishuai Jin & Yibin Fu: Axisymmetric necking versus Treloar-Kearsley instability in a hyperelastic sheet under equibiaxial stretching, *Math. Mech. Solids* **27** (2022), 1610-1631.

Electroelastic case

Yibin Fu & Xiang Yu: Axisymmetric necking of a circular electrodes-coated dielectric membrane, *Mechanics of Materials* **181** (2023), 104645.

1D reduced model

Xiang Yu & Yibin Fu : A 1D reduced model for the axisymmetric necking of a circular electrodes-coated dielectric membrane, to submit

Will focus on the purely mechanical case in order to simplify presentation.

1

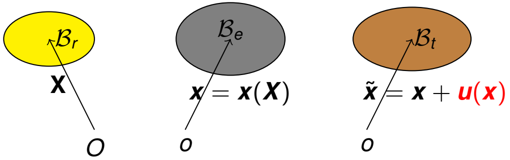
Governing equations

&

Linear analysis

Governing equations

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1}, \quad \text{Div}\, \boldsymbol{S} = \boldsymbol{0}.$$



Deformation gradient $\boldsymbol{F}$:

$$d\boldsymbol{x} = \bar{\boldsymbol{F}} d\boldsymbol{X}, \quad d\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{F}} d\boldsymbol{X}.$$

$$\Longrightarrow \quad \tilde{\boldsymbol{F}} = \bar{\boldsymbol{F}} + \eta \bar{\boldsymbol{F}}, \quad \text{where} \quad \eta = \text{grad}\, \boldsymbol{u}.$$

Eigenvalues of $\sqrt{\bar{\boldsymbol{F}}\bar{\boldsymbol{F}}^T}$ are $\lambda_1,\ \lambda_2,\ \lambda_3$. Incompressibility: $\lambda_1\lambda_2\lambda_3 = 1$.

Nominal stresses:

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(\bar{\boldsymbol{F}}, \bar{p}), \quad \tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}(\tilde{\boldsymbol{F}}, \tilde{p}).$$

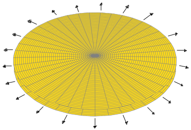
Convention: (r, θ, z) corresponds to $(1, 2, 3)$.

Primary deformation:

$$\bar{\boldsymbol{F}} = \lambda\, \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \lambda\, \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \lambda^{-2}\, \boldsymbol{e}_z \otimes \boldsymbol{e}_z,$$

and $\bar{p}$ is determined from $\bar{S}_{33} = 0$.

The bifurcation parameter is λ.



Incremental displacement: $\boldsymbol{u} = u(r, z)\, \boldsymbol{e}_r + v(r, z)\, \boldsymbol{e}_z$.

Define χ through

$$\chi^{\mathrm{T}} = \bar{\boldsymbol{F}}(\tilde{\boldsymbol{S}} - \bar{\boldsymbol{S}}), \quad p^* = \tilde{p} - \bar{p}.$$

We expand to obtain

$$\chi_{ij} = \mathcal{A}_{jilk}\eta_{kl} - p^*\delta_{ji} + (\bar{p} + p^*)(\eta_{ji} - \eta_{jm}\eta_{mi}) + \frac{1}{2}\mathcal{A}^2_{jilknm}\eta_{kl}\eta_{mn} + \cdots,$$

where

$$\boldsymbol{\eta} = \operatorname{grad}\boldsymbol{u} = \frac{\partial u}{\partial r}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{u}{r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \frac{\partial u}{\partial z}\boldsymbol{e}_r \otimes \boldsymbol{e}_z + \frac{\partial v}{\partial r}\boldsymbol{e}_z \otimes \boldsymbol{e}_r + \frac{\partial v}{\partial z}\boldsymbol{e}_z \otimes \boldsymbol{e}_z.$$

Equilibrium equation is $\operatorname{div}\chi^{\mathrm{T}} = \boldsymbol{0}$, i.e.

$$\chi_{1j,j} + \frac{1}{r}(\chi_{11} - \chi_{22}) = 0, \quad \chi_{3j,j} + \frac{1}{r}\chi_{31} = 0.$$

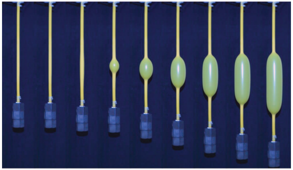
Incompressibility

$$\det(I + \boldsymbol{\eta}) = 0.$$

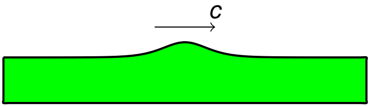
BCs: $\chi_{33} = \chi_{13} = 0$ on top and bottom surfaces.

Bifurcation condition

Methods used for two well-known localisation problems:



tube inflation



solitary water waves

Both involve a bifurcation from a uniform state into a homoclinic orbit

Both have translational invariance in the direction of localization

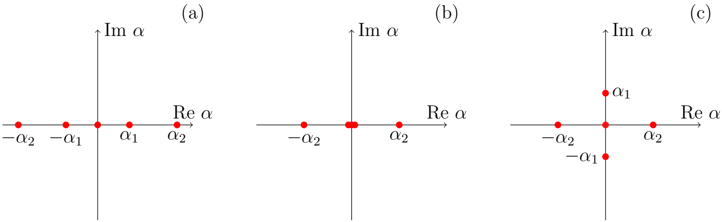
Such bifurcations can be analysed using

center manifold reduction (e.g. Kirchgässner 1988)
or
normal mode approach (e.g. Fu 2001)

Center manifold reduction:

$$\frac{\partial \boldsymbol{u}}{\partial x_1} = \mathcal{L}(\frac{\partial}{\partial x_2})\boldsymbol{u} = A\boldsymbol{u} + N(\boldsymbol{u}).$$

Look for a solution of the form $\boldsymbol{u}(x_1, x_2) = \boldsymbol{w}(x_2)\mathrm{e}^{\alpha x_1}$,
then localization takes place when 0 becomes a triple eigenvalue



However, current problem cannot be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial r} = \mathcal{L}(\frac{\partial}{\partial z})\boldsymbol{u}.$$

Bifurcation condition for localised necking

If

$$u = \frac{1}{r}\phi_z, \quad v = -\frac{1}{r}\phi_r,$$

then

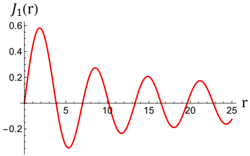
$$\alpha\left(\phi_{rrrr} - \frac{2}{r}\phi_{rrr} + \frac{3}{r^2}\phi_{rr} - \frac{3}{r^3}\phi_r\right) + 2\beta\left(\phi_{rrzz} - \frac{1}{r}\phi_{rzz}\right) + \gamma\phi_{zzzz} = 0,$$

where

$$\alpha = \mathcal{A}_{2323} > 0, \quad 2\beta = \mathcal{A}_{2222} + \mathcal{A}_{3333} - 2\mathcal{A}_{2233} - 2\mathcal{A}_{2332}, \quad \gamma = \mathcal{A}_{3232} > 0.$$

There is no translational invariance in the r-direction, hence use normal mode approach.

Look for a solution $\phi(r, z) = rJ_1(kr)S(kz)$.



The bifurcation condition can be factorised:

Bifurcation condition for <u>extensional</u> modes (A)

Bifurcation condition for <u>flexural</u> modes (B)

Expanding (A) to order $(kh)^2$, we obtain

$$\gamma(\beta+\gamma)+\frac{1}{24}\left\{\alpha\gamma-(2\beta+\gamma)^2\right\}(kh)^2+\cdots=0.$$

Conjecture: the bifurcation condition is

$$\gamma(\beta+\gamma)=0, \quad \Longrightarrow \quad \beta+\gamma=0.$$

Interpretation of the bifurcation condition

General biaxial tension of a membrane:

$$S_1 = \frac{\partial}{\partial \lambda_1} w(\lambda_1, \lambda_2), \qquad S_2 = \frac{\partial}{\partial \lambda_2} w(\lambda_1, \lambda_2), \quad S_3 = 0,$$

where $w(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$.

Equibiaxial tension ($\equiv$ all-round tension) corresponds to $\lambda_1 = \lambda_2 \equiv \lambda$.

The bifurcation condition for necking, $\beta + \gamma = 0$, is equivalent to

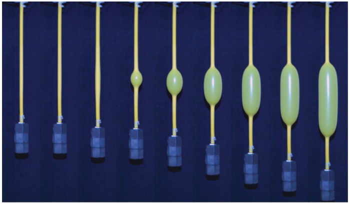
$$\left.\frac{\partial S_1(\lambda_1, \lambda_2)}{\partial \lambda_1}\right|_{\lambda_1=\lambda_2=\lambda} = 0.$$

Note that

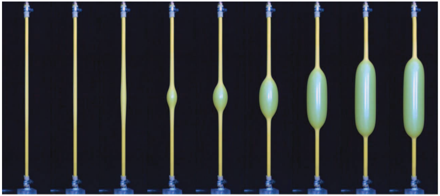
$$\text{LHS} \neq \frac{dS_1(\lambda, \lambda)}{d\lambda}.$$

Thus, the condition for necking does not coincide with the limit point in all-round tension.

Comparison with localized bulging in a rubber tube



fixed axial force



fixed ends/length

The bifurcation condition is $J(P, N) = 0$ where

$$J(P, N) = \begin{vmatrix} \frac{\partial P}{\partial v} & \frac{\partial P}{\partial \lambda_z} \\ \frac{\partial N}{\partial v} & \frac{\partial N}{\partial \lambda_z} \end{vmatrix}$$

which reduces to

$$\frac{dP}{dv} = 0 \text{ when N is fixed} \quad \text{or} \quad \frac{dN}{d\lambda_z} = 0 \text{ when P is fixed.}$$

For biaxial tension, we may compute

$$J(S_1, S_2) = \begin{vmatrix} \frac{\partial S_1}{\partial \lambda_1} & \frac{\partial S_1}{\partial \lambda_2} \\ \\ \frac{\partial S_2}{\partial \lambda_1} & \frac{\partial S_2}{\partial \lambda_2} \end{vmatrix}.$$

Does the bifurcation condition for localized necking correspond to $J(S_1, S_2) = 0$ at $\lambda_1 = \lambda_2$?

No, since the latter gives

$$\left(\frac{\partial S_1}{\partial \lambda_1} - \frac{\partial S_2}{\partial \lambda_1} \right)_{\lambda_1 = \lambda_2} = 0, \quad \text{or} \quad \left(\frac{\partial S_1}{\partial \lambda_1} + \frac{\partial S_1}{\partial \lambda_2} \right)_{\lambda_1 = \lambda_2} = \frac{dS_1(\lambda, \lambda)}{d\lambda} = 0.$$

$\downarrow$ TK instability

$\downarrow$ Limit point

TK instability = Treloar & Kearsley instability:

Treloar (1948, PRS), Kearsley (1986, IJSS)

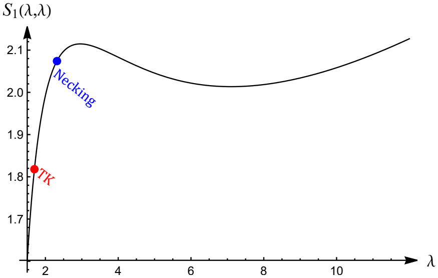
plane-strain version by Ogden (1985, IJSS)

A typical strain energy function admitting necking solutions:

$$W = 8\mu_1(\lambda_1^{1/2} + \lambda_2^{1/2} + \lambda_3^{1/2} - 3) + \frac{2\mu_2}{{m_2}^2}(\lambda_1^{m_2} + \lambda_2^{m_2} + \lambda_3^{m_2} - 3).$$

with $\mu_2 = \frac{1}{12}\mu_1, \; m_2 = 2$:

$$\lambda_{\text{necking1}} = 2.32, \quad \lambda_{\text{necking2}} = 7.49$$



2

Weakly nonlinear analysis

Weakly nonlinear analysis

From linear results we deduce that $k^2 \sim (\lambda - \lambda_{\rm cr})$ for small k, and

$$u = kJ_1(kr)S'(kz) \sim kJ_1(kr),$$

$$v = -\frac{1}{r}\left\{J_1(kr) + krJ_1'(kr)\right\} S(kz) \sim k^2 G(kr).$$

Thus, if $\lambda = \lambda_{\rm cr} + \epsilon\lambda_0$, then $k = O(\sqrt{\epsilon})$ and dependence on r is through $\xi = \sqrt{\epsilon}r$, and the near-critical solution takes the form

$$u = \sqrt{\epsilon}\left\{A(\xi) + \epsilon u^{(2)}(\xi, z) + \cdots\right\},$$

$$v = \epsilon\left\{v^{(1)}(\xi, z) + \epsilon v^{(2)}(\xi, z) + \cdots\right\}.$$

$$u = \sqrt{\epsilon}\left\{A(\xi) + \epsilon u^{(2)}(\xi, z) + \cdots\right\},$$

$$v = \epsilon\left\{v^{(1)}(\xi, z) + \epsilon v^{(2)}(\xi, z) + \cdots\right\}.$$

Amplitude equation:

$$\frac{d}{d\xi}\frac{1}{\xi}\frac{d}{d\xi}\xi P'(\xi) + c_1\lambda_0 P'(\xi) + c_2\frac{d}{d\xi}P^2(\xi) + c_3 A''(\xi)\left(A'(\xi) - \frac{1}{\xi}A(\xi)\right) = 0,$$

where $P(\xi)$ is defined by

$$P(\xi) = \frac{1}{\xi}(\xi A(\xi))' = \frac{2}{h}v^{(1)}(\xi, -\frac{h}{2}) \;\propto\; \text{deformed thickness}.$$

BCs:

$$\text{as} \quad \xi \to 0, \quad A(\xi), A''(\xi) \to 0$$

$$\text{as} \quad \xi \to \infty, \quad A(\xi) \to \frac{a_2}{\xi} - a_1 K_1(-c_1\lambda_0\xi).$$

Planar limit:

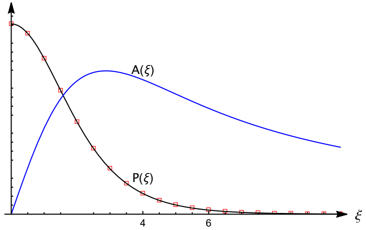
Setting all terms multiplied by $\frac{1}{\xi}$, $\frac{1}{\xi^2}$, etc to zero, we obtain

$$P'' + c_1\lambda^* P + \frac{1}{2}c_2^* P^2 = 0, \quad P = A'(\xi),$$

which has an explicit localised solution $P(\xi) \sim \mathrm{sech}^2(*)$.

General case:

Finite difference solution:
(initial guess = planar solution divided by $1 + \xi^2$)



Squares given by

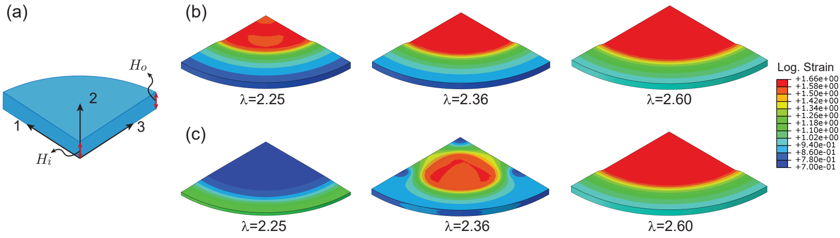
$$P(\xi) = \frac{a}{b\,\xi^2 + 1}\mathrm{sech}^2(c\,\xi).$$

3

Abaqus simulations

Fully nonlinear numerical simulations

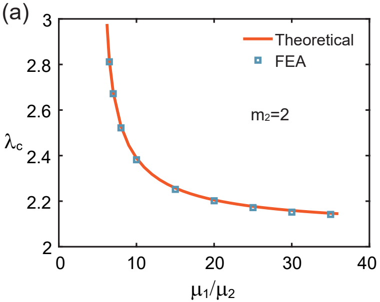
Imperfections at the centre or edge:



$$W = \frac{2\mu_1}{m_1^2}(\lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}} + \lambda_3^{\frac{1}{2}} - 3) + \frac{\mu_2}{8}(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 3),$$

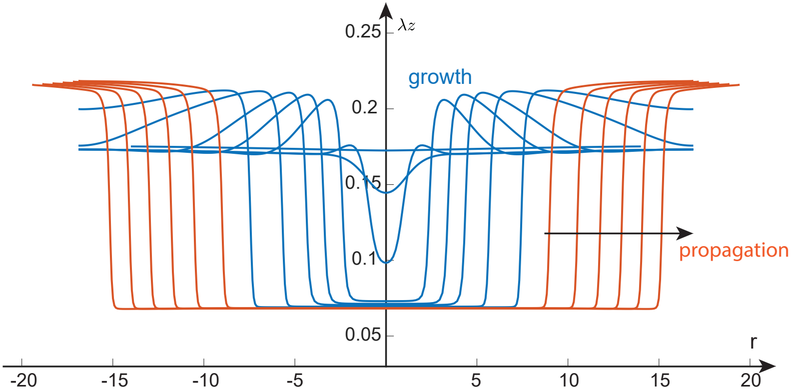
with $\mu_2/\mu_1 = 1/80$.

Comparison between theory and simulations:



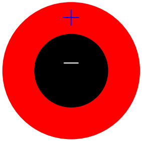
$$W = 8\mu_1(\lambda_1^{1/2} + \lambda_2^{1/2} + \lambda_3^{1/2} - 3) + \frac{2\mu_2}{{m_2}^2}(\lambda_1^{m_2} + \lambda_2^{m_2} + \lambda_3^{m_2} - 3).$$

Initiation $\rightarrow$ growth $\rightarrow$ propagation (Maxwell state):



A typical Abaqus simulation video

Necking propagation (Maxwell state)



- transition region replaced by a sharp interface at $R = R_i$
- λ_z^- and λ_z^+ are both constant

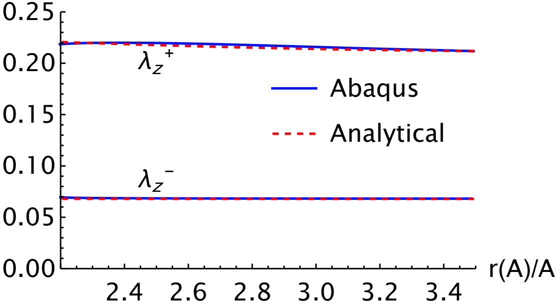
A uniform circular region ("$-$ phase") surrounded by an annular outer region ("$+$ phase").

Total energy:

$$\mathcal{E} = \frac{1}{2} R_i^2 w\Big(\frac{1}{\sqrt{\lambda_z^-}}, \frac{1}{\sqrt{\lambda_z^-}}\Big) + \int_{R_i}^{A} w\Big(\lambda_1, \frac{1}{\lambda_1 \lambda_z^+}\Big) R dR - PA^2 \lambda_1 |_{R=A}.$$

Require $\mathcal{E}$ to be stationary w.r.t. λ_z^-, λ_z^+ and R_i:

$$\frac{\partial \mathcal{E}}{\partial \lambda_z^-} = 0, \quad \frac{\partial \mathcal{E}}{\partial \lambda_z^+} = 0, \quad \frac{\partial \mathcal{E}}{\partial R_i} = 0.$$


0.25
0.20
0.15
0.10
0.05
0.00
λ_z^+
λ_z^-
Abaqus
Analytical
2.4
2.6
2.8
3.0
3.2
3.4
r(A)/A

4

1D reduced model

1D reduced model

Variational-asymptotic method first proposed by Berdichevskii (1979). See also Audoly & Hutchinson (JMPS, 2016), ...

Homogeneous solution: $r = \mu R, \quad z = \lambda Z$, and so

$$\lambda_1 = \lambda_2 = \mu, \quad \lambda_3 = \lambda.$$

At least in the early stage of necking formation, we may assume that

$$r = \mu(S)R, \quad z = \lambda(S)Z, \quad S = \epsilon R, \qquad (*)$$

where $\lambda(S)$ is found from the incompressibility condition.

For a consistent solution, we add correction terms:

$$\begin{aligned} r(R,Z) &= \varepsilon^{-1}\mu(S)S + \varepsilon u^*(S,Z) + O(\varepsilon^3), \\ z(R,Z) &= \lambda(S)Z + \varepsilon^2 v^*(S,Z) + O(\varepsilon^4). \end{aligned}$$

Step 1: Fix μ and minimize w.r.t. u^* and v^*. The result is

$$\mathcal{E}_{1d}[\mu] = \int_0^A L(R, \mu, \mu', \mu'')dR,$$

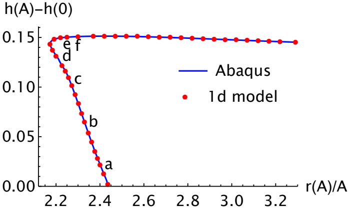
where

$$L(R, \mu, \mu', \mu'') = \left(w(\lambda, \mu) + \frac{H^2 w_1(\lambda, \mu)}{24\lambda} \lambda'(R)^2 \right) R.$$

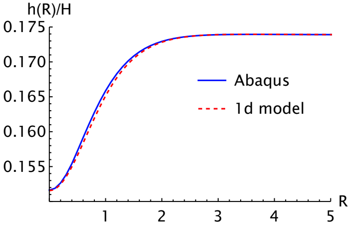
Step 2: Minimize w.r.t. μ; the associated Euler–Lagrange equation then yields the 1D model:

$$\frac{\partial L}{\partial \mu} - \frac{d}{dR}\left(\frac{\partial L}{\partial \mu'}\right) + \frac{d^2}{dR^2}\left(\frac{\partial L}{\partial \mu''}\right) = 0.$$

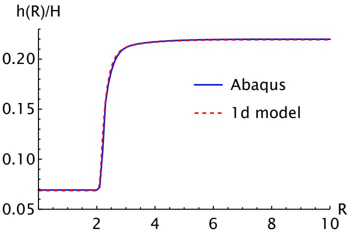
This equations recovers the bifurcation condition and the weakly nonlinear theory exactly.



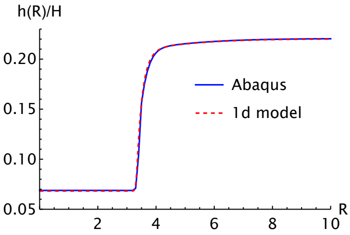
Bifurcation diagram



(a)



(e)



(f)

Conclusion

- Axisymmetric necking analysed for the first time;
- The entire necking process, from initiation, growth, to propagation, can be described analytically or semi-analytically;
- Bifurcation condition for axisymmetric necking derived, not given by $J(S_1, S_3) = 0$;
- Near-critical analysis conducted, amplitude equation solved using FD;
- A 1D model derived, with predictions in excellent agreement with Abaqus simulations;
- Current work: experimental verification.

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Conclusion

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