Collisions in fluid/solid mixtures

Matthieu Hillairet

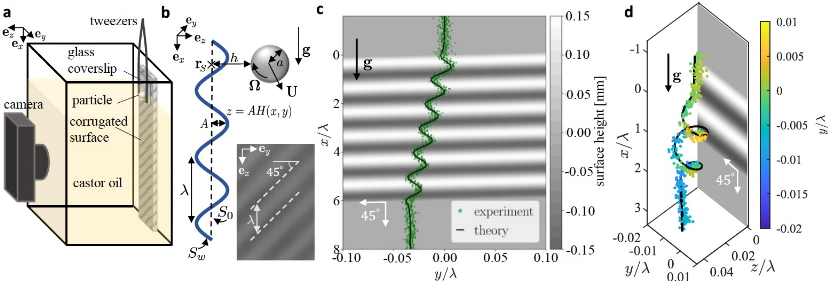
(IMAG, Université de Montpellier)

MPDE'24, Prague

Motivation

Close-to-contact motion of particles

Example 1. Fall a particle along a corrugated wall



From : D.L. Chase, C. Kurzthaler, H.A. Stone, PNAS ’22

Example 2 : Effective properties of fluid/particle mixtures.

No particle With particles

From the webpage of Daniel D. Joseph

Ref : https://dept.aem.umn.edu/~./faculty/joseph/nano_siphons/index.shtml

Context.

Particles in a fluid

Evolution system.

- Favorite fluid equation
  - Stokes / Navier-Stokes / Euler
  - Compressible / Incompressible
  - Newtonian / Non-newtonian
- Newton solid mechanics

$$\frac{\mathrm{d}}{\mathrm{d}t}[mV] = -\int_{\partial B(t)} \Sigma_s n \mathrm{d}\sigma$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbb{J}\omega] = -\int_{\partial B(t)} (x - G) \times \Sigma_s n \mathrm{d}\sigma$$

- Coupling
  - Mandatory :
    no-penetration + continuity of stress tensor
  - Open : slip/no-slip condition

A 2D example

Navier Stokes equations with no slip boundary conditions

Unknowns : (v, p) and $(G_i, \theta_i)_{i=1,\dots,N}$ Parameters : $\rho, \mu, m_i, \mathbb{J}_i$

System :

$$\left.\begin{aligned} \rho\left(\partial_t v + v\cdot\nabla v\right) &= \mu\Delta v - \nabla p \\ \operatorname{div} v &= 0 \end{aligned}\right\} \quad \text{on } \mathcal{F}(t) = \Omega\backslash\overline{\bigcup_{i=1}^{N} B_i(t)}$$

where $B_i(t) = G_i(t) + R_{\theta_i(t)} B_i^0$

$$\begin{cases} v(t,x) = V_i(t) + \omega_i\,(x - G_i)^{\perp}\,, & \text{on } \partial B_i(t), \\ v(t,x) = w, & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} m_i \dot{V}_i &= -\displaystyle\int_{\partial B_i(t)} (2\mu D(v) - p\mathbb{I}_2)\, n\mathrm{d}\sigma \\ \dot{G}_i &= V_i \\ \mathbb{J}_i\dot{\omega}_i &= -\displaystyle\int_{\partial B_i(t)} (x - G_i)^{\perp}\cdot(2\mu D(v) - p\mathbb{I}_2)\, n\mathrm{d}\sigma \\ \dot{\theta}_i &= \omega_i \end{cases}$$

Issue : $\min\{d(B_i(t)B_j(t)), d(B_i(t), \partial\Omega), i \neq j\} = 0\,??$

A 2D example

Navier Stokes equations with no slip boundary conditions

Unknowns : (v, p) and $(G_i, \theta_i)_{i=1,\dots,N}$ Parameters : $\rho, \mu, m_i, \mathbb{J}_i$

System :

$$\left.\begin{aligned} \rho\left(\partial_t v + v \cdot \nabla v\right) &= \mathrm{div}\Sigma(v, p) \\ \text{div } v &= 0 \end{aligned}\right\} \quad \text{on } \mathcal{F}(t) = \Omega \backslash \overline{\bigcup_{i=1}^{N} B_i(t)}$$

where $B_i(t) = G_i(t) + R_{\theta_i(t)} B_i^0$

$$\begin{cases} v(t,x) = V_i(t) + \omega_i \left(x - G_i\right)^{\perp}, & \text{on } \partial B_i(t), \\ v(t,x) = w, & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} m_i \dot{V}_i &= -\displaystyle\int_{\partial B_i(t)} \Sigma(v,p)\, n \mathrm{d}\sigma \\ \dot{G}_i &= V_i \\ \mathbb{J}_i \dot{\omega}_i &= -\displaystyle\int_{\partial B_i(t)} (x - G_i)^{\perp} \cdot \Sigma(v,p)\, n \mathrm{d}\sigma \\ \dot{\theta}_i &= \omega_i \end{cases}$$

Issue : $\min\{d(B_i(t) B_j(t)), d(B_i(t), \partial\Omega), i \neq j\} = 0\,??$

Construction of solutions

Euler equations :

- incompressible

JG Houot, J. San Martin, M. Tucsnak '10

- compressible

In progress ? ?

Navier Stokes

- incompressible

C. Grandmont & Y. Maday '98, K.H. Hoffmann & V. Starovoitov '99, B. Desjardins & M. Esteban 99' ; B. Desjardins & M. Esteban '00, M.D. Gunzburger, H.-C. Lee & G. A. Seregin '00, J. A. San Martin, V. Starovoitov & M. Tucsnak '02, E. Feireisl '02, T. Takahashi '03 , M. Tucsnak & T. Takahashi '04, S. Nečasová '09, D. Gérard-Varet, M.H. ' 13, M. Geissert, K. Götze, & M. Hieber '14,

- compressible

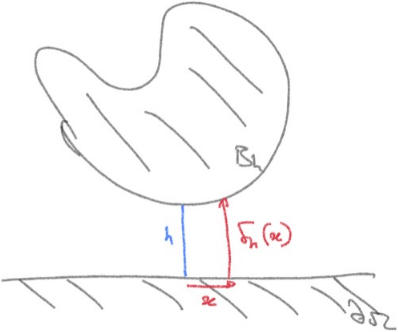
E. Feireisl '03, B. H. Haak, D. Maity, T. Takahashi & M. Tucsnak '19, M. Hieber & M. Murata '19, S. Nečasová, M. Ramaswamy, A. Roy & A. Schlömerkemper '22

(Non)-uniqueness. O. Glass & F. Sueur '13, V.N. Starovoitov '05.

Collision issue. F. Oschmann '24

Tackling collision issue

A simple 2D pre-collisional geometry



Vertical motion of B above a ramp

Assumptions

- One solid B
- $\Omega = \mathbb{R}^2_+$
- Only vertical motion

Notations

- $h = \mathrm{dist}(B, \partial\Omega)$
- $B = B^h$
- $\delta_h(x)$ vertical distance

Two cases $x \in (-\lambda_*, \lambda_*)$

- Symmetric :

$$\delta_h(x) = h + \kappa|x|^{1+\alpha} \qquad \alpha \geq 0$$

- General : $\delta \in C^\infty$

$$\delta_h'(0) = 0 \quad \delta_h(0) = h.$$

Tackling collision issue

Starovoitov's argument

Lemma. Assume that $w \in [W_0^{1,p}(\Omega)]^2$ satisfies

- $w = \ell e_2$ on B^h
- $\mathrm{div}\, w = 0$ on Ω

Then there exists C (independent of w, h) s.t :

$$|\ell| \leq C h^{\frac{1+2\alpha}{1+\alpha} - \frac{2+\alpha}{1+\alpha}\frac{1}{p}} \|\nabla w\|_{L^p(\Omega)}.$$

Proof : Integrate $\mathrm{div}\, w = 0$ in the gap below B^h.

Remark.
If $\alpha = 1$ exponant is larger than 1 iff $p \geq 3$
If $p = 2$ exponent is larger than 1 iff $\alpha \geq 2$.

Applications.
J.L. Vázquez and E. Zuazua '06 , E. Feireisl, M.H., S. Nečasová, '08

Extensions. S. Filippas and A. Tersenov '21.

Lorentz reciprocal formula

Lemma. Let (w,q) and $(\tilde{w},\tilde{q})$ (say $H^2(\mathcal{F})\times H^1(\mathcal{F})$) and $\ell\in\mathbb{R}$ such that :

- $w=\tilde{w}=0$ on $\partial\Omega$, and $w=\ell e_2$, $\tilde{w}=e_2$ on ∂B^h
- $\mathrm{div}\, w=\mathrm{div}\,\tilde{w}=0$ on $\Omega\setminus\overline{B}$

there holds :

$$\int_{\partial B}\Sigma(w,q)n\cdot e_2 \mathrm{d}\sigma-\ell\int_{\partial B^h}\Sigma(\tilde{w},\tilde{q})n\cdot e_2\mathrm{d}\sigma = \int_{\Omega\setminus\overline{B^h}}((\Delta w-\nabla q)\cdot\tilde{w}-(\Delta\tilde{w}-\nabla\tilde{q})\cdot w)$$

Application. (symmetric case)

$$\tilde{w}(x,y)=\nabla^{\perp}\phi\,,\qquad \phi(x,y)=\left[xP^{(1)}_{opt}\left(\frac{y}{\delta_h(x)}\right)\right]\,,\qquad P^{(1)}_{opt}(t)=3t^2-2t^3.$$

$$\Delta\tilde{w}(x,y)=\begin{pmatrix}-\partial_{211}\phi_h-\partial_{222}\phi_h\\ \partial_{111}\phi_h+\partial_{122}\phi_h\end{pmatrix}$$

with :

$$\partial_{222}\phi(x,y)=-\frac{16x}{\delta_h(x)^3}=\partial_1\tilde{q}(x)$$

References

Multiplier methods.
T.I. Hesla ’05
M.H. ’07, M.H. and T. Takahashi ’09’10’21, D. Gérard-Varet and M.H. ’10,
D. Gérard-Varet, M.H. and C. Wang ’15
L. Sabbagh ’19, L. Sabbagh and M.H. ’23
B. J. Jin, S. Nečasová, F. Oschman, A. Roy ’23
S. Nečasová and F. Oschman ’23
D. Bonheure, M.H., G. Sperone and C. Patriarca ’24

Fluid-structure interactions. C. Grandmont & M.H. ’16

Asymptotics of Stokes problem.
J. Happel and H. Brenner ’65
M.D.A Cooley O’Neill ’68’69, L.M. Hocking ’73, R.G. Cox ’74,
D. Gérard-Varet & M.H. ’12, M.H. & T. Kelaï ’15.
Y. Gorb ’16,
H. Ammari, H. Kang, D. Kim, S. Yu ’23
H. Li and L. Xu ’23, H. Li, L. Xu and P. Zhang ’23,’24

Potential flows.
J.-G. Houot and A. Munnier ’08, A. Munnier and K. Ramdani ’15

Computing Stokes asymptotics

M.H., A. Lozinski, M. Szopos, '11 ; D. Gérard-Varet & M.H. '12

Lemma. Let $h > 0$

The "unique" pair $(w, q) \in H^2(\mathbb{R}^2_+ \setminus \overline{B^h}) \times H^1(\mathbb{R}^2_+ \setminus \overline{B^h})$ solution to :

$$\left\{ \begin{array}{rcl} \mu\Delta w - \nabla q & = & 0 \\ \mathrm{div}\, w & = & 0 \end{array} \right. \text{ in } \Omega \setminus \overline{B^h}, \quad \left\{ \begin{array}{rcl} w_{|_{\partial\Omega}} & = & 0 \\ w_{|_{\partial B^h}} & = & e_2 \end{array} \right.$$

is characterised by :

$$w = \mathrm{argmin} \underbrace{\left\{ \int_{\mathbb{R}^2_+} |D(u)|^2, \ u \in C^\infty_c(\mathbb{R}^2_+) \text{ with } \mathrm{div}\, u = 0 \text{ and } u|_{\partial B^h} = e_2 \right\}}_{Y_h}.$$

Moreover :

$$\int_{\partial B^h} \Sigma(w, q) n \mathrm{d}\sigma \cdot e_2 = 2 \int_{\Omega \setminus \overline{B^h}} |D(w)|^2.$$

The method "of reduced functionals"

Framework. We want to minimize $\mathcal{E}$ on the space Y and we have :

There exist two mappings and a functional $\tilde{\mathcal{E}}$

$$\Psi : Y \to \tilde{Y} \quad \mathbf{V} : \tilde{Y} \to Y$$

such that

$$\tilde{\mathcal{E}}(\Psi(\mathbf{v})) \leq \mathcal{E}(\mathbf{v}) \quad \forall \mathbf{v} \in Y.$$

Lemma.

Assume $\tilde{\mathcal{E}}$ has a minimum that is reached in ψ_0, then there holds :

$$\inf_{\tilde{Y}} \tilde{\mathcal{E}}(= \tilde{\mathcal{E}}(\psi_0)) \leq \inf_{Y} \mathcal{E} \leq \mathcal{E}(\mathbf{V}[\psi_0]).$$

Application.

Rewriting of Munnier-Ramdani result [M.H., D. Seck, L. Sokhna '18]

Stokes asymptotics - Symmetric case.

Any $u \in Y_h$ reads $u = \nabla^{\perp}\phi$ with :

$$(*) \qquad \left\{ \begin{array}{rcl} \nabla\phi(x,\delta_h(x)) & = & e_1 \\ \phi(x,0) = \partial_2\phi(x,0) & = & 0 \end{array} \right.$$

Energy to minimize :

$$\mathcal{E}_h := \int_{\mathcal{F}^h} |\nabla u(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

Restriction operator : $\Psi = \phi_{|\mathcal{G}}$ where $\mathcal{G} = \{|x| < \lambda_*, y \in (0,\delta_h(x))\}$

$$\tilde{Y}_h := \left\{ \phi \in H^2(\mathcal{F}_0^h) \ \text{ s.t. } \ (*) \right\} \qquad \tilde{\mathcal{E}}_h := \int_{\mathcal{F}_0^h} |\partial_{22}\phi(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

Extension operator : $V(\phi) = v$

$$v = \nabla^{\perp}(\zeta_{\mathcal{G}}\phi + (1-\zeta_{\mathcal{G}})\zeta_0(x,y)(x-c))$$

where $\zeta_{\mathcal{G}}$ truncates in $\mathcal{G}$ horizontally, ζ_0 depends only on B.

The method "of reduced functionals"

Application

Profile of the approximate minimizer – Symmetric case

$$\phi_h(x,y) = xP^{(1)}_{opt}\left(\frac{y}{\delta_h(x)}\right) \qquad P^{(1)}_{opt}(t) = 3t^2 - 2t^3$$

$$\tilde{\mathcal{E}}(h) = \|P^{(1)}_{opt}\|^2_{L^2((0,1))} \int_{-\lambda_*}^{\lambda_*} \frac{x^2}{(h+\kappa|x|^{1+\alpha})^3}\mathrm{d}x.$$

Example : Shape sensitivity of the drag [D.Gérard-Varet & M.H. '12]

There holds :

$$\tilde{\mathcal{E}}_h(x) \sim C_{\alpha,\kappa} h^{-3\alpha/(1+\alpha)}$$

Threshold divergence $1/h$ (= existence of collision) when $\alpha = 1/2$

Remark.

$$\tilde{\mathcal{E}}(h) = \partial_h \mathcal{E}_{col}(h)\,, \qquad \mathcal{E}_{col}(h) = \int_{-\lambda_*}^{\lambda_*} \frac{x^2}{(h+\kappa|x|^{1+\alpha})^2}\mathrm{d}x.$$

The method "of reduced functionals"

Application

Stokes asymptotics - General case. $\phi_* \in (x,y) \mapsto \{x^2 + y^2, x, y\}$

Any $u \in Y_h$ reads $u = \nabla^{\perp}\phi$ with :

$$(*) \qquad \left\{ \begin{array}{rcl} \nabla\phi(x,\delta_h(x)) & = & \nabla\phi_*(x,\delta_h(x)) \\ \phi(x,0) = \partial_2\phi(x,0) & = & 0 \end{array} \right.$$

Profile of the approximate minimizer

$$\phi_h(x,y) = (\phi_*(x,\delta_h(x)) - c_*)P^{(1)}_{opt}\left(\frac{y}{\delta_h(x)}\right) + \partial_2\phi_*(x,\delta_h(x))\partial_x\delta_h(x)P^{(2)}_{opt}\left(\frac{y}{\delta_h(x)}\right)$$

with $P^{(1)}_{opt}(t) = 3t^2 - 2t^3$, $P^{(2)}_{opt}(t) = t^2(t-1)$ and c_* prescribed by :

$$\int_{-\lambda^*}^{\lambda_*} \partial_{222}\phi_h(x,0)dx = 0$$

Associated pressure. $p_h(x,y) = \partial_{222}\phi_h - \partial_{12}\phi_h$.

An elliptical object in a 2D-channel

with D. Bonheure (ULB), G. Sperone (ULB/Polimi) and C. Patriarca (ULB/Polimi)

Setting

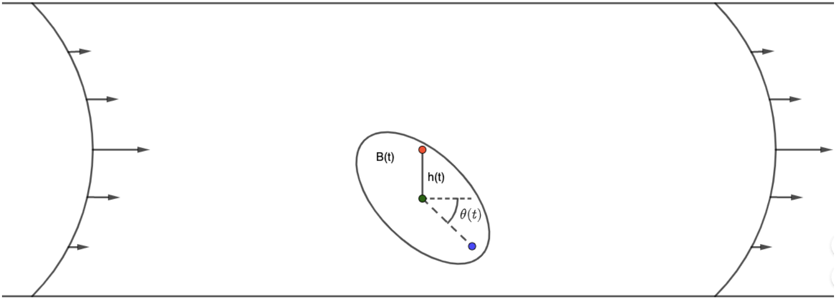


FIGURE - Ellipse in a channel

Issue : Does the elliptical object come back to rest position ?

An elliptical object in a 2D-channel

System

$$\begin{cases} \partial_t u + u \cdot \nabla u = \Delta u - \nabla p, \\ \mathrm{div}\, u = 0 \end{cases} \quad \text{in } \Omega_v\,,$$

$$\begin{cases} u = \lambda_0 v_p(x_2) = \lambda_0 (1 - x_2^2) e_1 & \text{on } |x_1| = \infty \\ u = 0 & \text{on } \Gamma\,, \\ u = h'\, e_2 + \theta' (x - h e_2)^\perp & \text{on } \partial B(t)\,, \end{cases}$$

$$\begin{cases} m\ddot{h} + \partial_1 F(h, \theta) = -e_2 \cdot \displaystyle\int_{\partial B(t)} \Sigma(u,p) n \, d\sigma\,, \\ \mathcal{J} \ddot{\theta} + \partial_2 F(h, \theta) = - \displaystyle\int_{\partial B(t)} (x - h e_2)^\perp \cdot \Sigma(u,p) n \, d\sigma \end{cases}$$

Cauchy theory. If the amplitude of v_p is small, we have existence of a weak solution satisfying

$$\frac{1}{2} \left[\int_{\mathcal{F}(t)} |u - v_p|^2 + m |\dot{h}|^2 + \mathcal{J} |\dot{\theta}|^2 + F(h, \theta) \right] + 2 \int_0^t \int_{\mathcal{F}(s)} |D(u - v_p)|^2 \leq C_0 t$$

as long as there is no collision between $B(t)$ and Γ.

Main result

Theorem For any compatible initial data
$((h_0,\theta_0),u_0,(h_0',\theta_0')) \in A \times \{L^2(\Omega_0)+v_p\} \times \mathbb{R}^2$, there exists $p_*^{(1)} > 0$ such that, if $p_0 \leq p_*^{(1)}$, then

(i) there exists a global-in-time weak solutions (U,h,θ);

(ii) there exists a constant $\mathfrak{d}^0_{min} > 0$, such that

$$\mathrm{d}(B(t),\partial A) \geq \mathfrak{d}^0_{min}\,, \qquad \forall t \geq 0\,.$$

Theorem For any compatible initial data
$((h_0,\theta_0),U_0,(h_0',\theta_0')) \in A \times \{L^2(\Omega_0)+v_p\} \times \mathbb{R}^2$, there exists $p_*^{(2)} \in (0,p_*^{(1)}]$ such that, if $p_0 \leq p_*^{(2)}$ and (U,h,θ) is a global weak solution then (U,h,θ) converges to the equilibrium $(U_{\mathrm{eq}},0,0)$ as $t\to\infty$ in the following sense :

$$\lim_{t\to\infty} \|U(t)-U_{\mathrm{eq}}\|^2_{L^2(A)} + (|h(t)|^2+|\theta(t)|^2) + (|h'(t)|^2+|\theta'(t)|^2) = 0\,.$$

Distance estimate

Claim. If the amplitude of the Poiseuille flow is small, then there si a minimal distance $\delta_0 > 0$ between $B(t)$ and Γ uniformly in time.

Rough computation. The potential energy of collision

$$\mathcal{E}_{col}(h,\theta) = \frac{1}{2}\int_{-\lambda_*}^{\lambda_*} \frac{x^2 \mathrm{d}x}{(h+\kappa_\theta x^2)^2}$$

satisfies :

$$\dot{\mathcal{E}}_{col}(h,\theta) = -\dot{h}\int_{-\lambda_*}^{\lambda_*} \frac{x^2 \mathrm{d}x}{(h+\kappa_\theta x^2)^3} - \dot{\theta}\partial_\theta \kappa_\theta \underbrace{\int_{-\lambda_*}^{\lambda_*} \frac{x^4 \mathrm{d}x}{(h+\kappa_\theta x^2)^3}}_{Rem}$$

with

$$|Rem| \sim \frac{C}{\sqrt{h}} \sim |\mathcal{E}_{col}(h,\theta)|$$

Challenge. γ_θ is time-dependent and is not symmetric.

- To prove decay in energy estimate.
- To interpret $\lim_{h\to 0}\sqrt{h}Rem$ as a function of θ.

Return to rest

Structure model equation.

$$m\ddot{h} + \kappa h = -\dot{h}$$

Estimates.

- multiply by $\dot{h}$

$$\frac{\mathrm{d}}{2\mathrm{d}t}\left[m|\dot{h}|^2 + \kappa|h|^2\right] + |\dot{h}|^2 = 0$$

- multiply by h

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{|h|^2}{2} + m\dot{h}h\right] + \kappa|h|^2 - m|\dot{h}|^2 = 0.$$

Conclusion. Exponential decay of a functional

$$\mathcal{F}(h) = \alpha_1|h|^2 + \alpha_2|\dot{h}|^2$$

Return to rest

Real case.

$$m\ddot{h} + \partial_1 F(h,\theta) = -\int_{\partial B} \Sigma(u,p) n d\sigma \cdot e_2$$

$$J\ddot{\theta} + \partial_2 F(h,\theta) = -\int_{\partial B} (x - he_2)^{\perp} \cdot \Sigma(u,p) n d\sigma$$

Perturbed Navier Stokes

Estimates.

- multiply by $(\dot{h}, \dot{\theta}, u)$
- multiply by h, θ and lifting of $(he_2 + \theta(x - he_2)^{\perp})$

Conclusion. Eventually, total energy $\lesssim C\lambda_0$.

Thank you for your attention.