Regularity and Asymptotic behavior of volume preserving geometric flows

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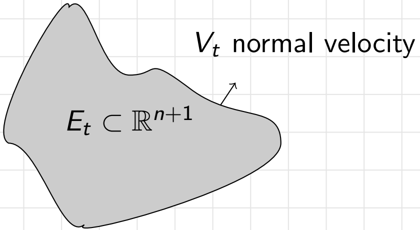
Definition

Mean curvature flow (MCF) for evolution of sets $(E_t)_{t\geq 0}$

$$V_t = -H_{E_t} \qquad \text{on } \partial E_t.$$

Q: Don't we know everything about it? Maybe yes, if E_0 is mean convex/no fattening (Hamilton, Huisken, White, Ilmanen, ...).

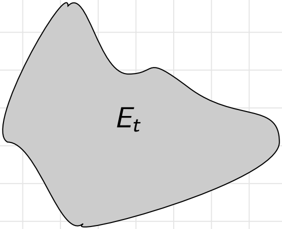


Definition

Family of sets $(E_t)_{t\in[0,T)}$ is a classical solution to the MCF starting from $E_0 \subset \mathbb{R}^{n+1}$ if there is a smooth family of diffeos $(\Phi_t)_{t\in[0,T)}$ such that $\Phi_0 = id$, $E_t = \Phi_t(E_0)$ and

$$V_t = -H_{E_t} \qquad \text{on } \partial E_t.$$

Gradient flow of the surface area: Decrease the perimeter of E_0 continuously as fast as possible. To measure the continuity we need to fix the metric.



Gradient flow of the surface area: Decrease the surface area of E_0 continuously as fast as possible. To measure the continuity we need to fix the metric. If we choose

- L^2-norm, we get the MCF

$$V_t = -H_{E_t}$$

- H^{-1}-norm we get the surface diffusion

$$V_t = \Delta_{\partial E_t} H_{E_t}$$

- $H^{-1/2}$-norm we get the Mullins-Sekerka

$$V_t \simeq \Delta_{\partial E_t}^{1/2} H_{E_t}$$

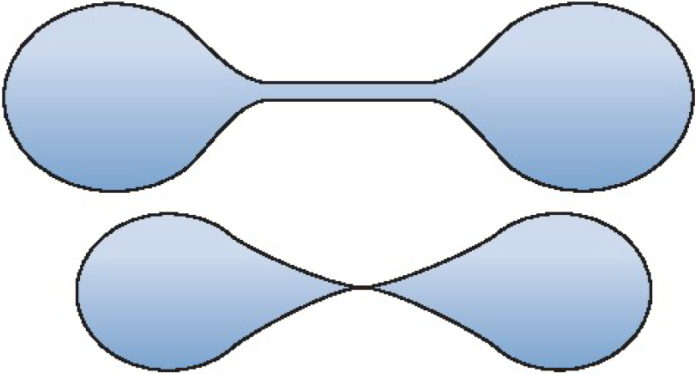
- L^2-norm with volume constraint, we get the Volume preserving mean curvature flow (VMCF)

$$V_t = -H_{E_t} + \bar{H}_{E_t}$$

Singularity

The classical solution $(E_t)_{t\in[0,T)}$ may form singularity: E_T is not diffeomorphic to $E_0 \Rightarrow T$ is a singular time.

This is either because E_T is no longer homeomorphic to E_0 or because of lack of regularity.



Weak solution

In order to continue the flow over the singular time we need to

- Operate a surgery (Huisken-Sinestrari, Huisken-Brendle)
- Consider a weak solution which is defined for all times

Possible notions of weak solutions for the MCF

- Brakke-flow (Brakke 78').
- Level-set solution via viscosity theory (Chen-Giga-Goto 89', Evans-Spruck 91'). Needs comparison principle: $E_0 \subset F_0 \;\Rightarrow E_t \subset F_t$
- Distributional solution (e.g. Luckhaus-Stürzenhecker 95')

Weak solution for MCF: $V_t = -H_{E_t}$

If E_t is a smooth solution to the MCF and $\varphi \in C^1$ then

$$\frac{d}{dt}\int_{\partial E_t} \varphi \, d\mathcal{H}^n = \int_{\partial E_t} (-H_{E_t}^2 \varphi + \partial_t \varphi + H_{E_t} \nabla \varphi \cdot \nu_{E_t}) \, d\mathcal{H}^n.$$

Brakke solution: Replace $'' =''$ with $'' \leq ''$ and ∂E_t with varifold M_t

Distributional solution: $\varphi \in C_0^1$, multiply the equation by φ and integrate by parts

$$\int_0^\infty \int_{E_t} \partial_t \varphi \, dxdt = \int_0^\infty \int_{\partial E_t} H_{E_t} \varphi \, d\mathcal{H}^n dt.$$

Methods to obtain the weak solution

- Viscosity solution via Perron's method
- Phase-field approximation, MBO thresholding scheme,...
- Flat flow via minimizing movements scheme

Weak solution by minimizing movement scheme:
Idea is to apply the implicit Euler method: Gradient flow for the function $F : \mathbb{R}^n \to \mathbb{R}$

$$\frac{d}{dt}x(t) = -\nabla F(x(t)).$$

Fix a time step $h > 0$ and initial point $x_0^h = x_0$. For k choose the next point x_{k+1}^h as minimum of the function

$$F(x) + \frac{|x - x_k^h|^2}{2h}.$$

$$\Rightarrow \frac{x_{k+1}^h - x_k^h}{h} = -\nabla F(x_{k+1}^h).$$

Define $x^h(t) = x_k^h$ for $\frac{t}{h} \in [k, k+1)$. Passing $h \to 0$ we get the solution. (Why?)

Weak solution to the VMCF by minimizing movement scheme :

Fix a time step $h > 0$ and initial set $E_0^h = E_0 \subset \mathbb{R}^{n+1}$. For k choose the next set E_{k+1}^h as (any) minimum of the functional

$$\min\left\{P(E) + \frac{1}{h}\int_E \bar{d}_{E_k^h}\,dx : |E| = |E_0|\right\}$$

Euler-Lagrange equation

$$\frac{\bar{d}_{E_k^h}}{h} = -H_{E_{k+1}^h} + \lambda_{k+1}^h \qquad \text{on } \partial E_{k+1}^h.$$

Define $E_t^h = E_k^h$ for $\frac{t}{h} \in [k, k+1)$, which is called the approximative flow. Any cluster point as $h \to 0$ is **a flat flow**. It is well defined for all times. Formally it solves

$$V_t = -H_{E_t} + \bar{H}_{E_t}$$

Flat flow for

- MCF (Almgren-Taylor-Wang, Luckhaus-Stürzenhecker 95')
- Mullins-Sekerka (Luckhaus-Stürzenhecker 95')
- VMCF (Mugnai-Seis-Spadaro 2016).
- Surface diffusion (Cahn-Taylor 94') but the existence is not known.

I will mostly concentrate on VMCF.

Questions:

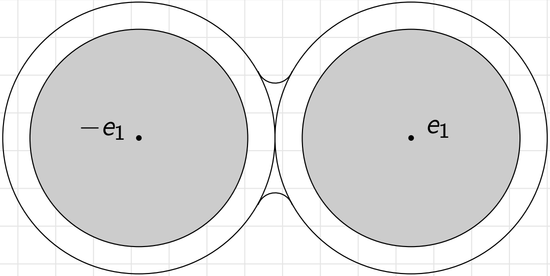
- Uniqueness
- Partial regularity
- Does the flat flow solve the associated PDE in a weak sense?
- Asymptotic behavior

Comment on uniqueness:

In general, there can be many weak solutions even for

$$V_t = -H_{E_t}.$$

This is called **the fattening phenomenon**. For example, choose the initial $E_0 = B_1(-e_1) \cup B_1(e_1)$ (Fusco-Julin-Morini (2020), Bellettini-Paolini (2002))



Regularity/Consistency: Consider the VMCF

$$V_t = -H_{E_t} + \bar{H}_{E_t}.$$

If the initial set E_0 is regular, say C^2, is the flat flow regular for a short time? If yes, this implies consistency since smooth solutions are unique.

Difficult: Recall that we got the flat flow as a limit of the approximating flow (E_t^h) as the time step $h \to 0$. We do not know that the limit solves any equation.

Idea: Prove regularity estimates for (E_t^h) which are uniform in h.

$$V_t = -H_{E_t} + \bar{H}_{E_t} \qquad \text{VMCF}$$

Theorem (Julin-Niinikoski 2022)

Assume $E_0 \subset \mathbb{R}^{n+1}$ satisfies interior and exterior ball condition (UBC) radius r_0. Then for $r < r_0$ there is T such that (E_t^h) satisfies UBC with radius r for all $t \leq T$ and all $h \leq h_0$. This condition is open in the sense that if E_t^h satisfies UBC with radius r for all $t \leq T$, then there is $\delta > 0$ s.t. it satisfies UBC with radius $r/2$ for all $t < T + \delta$.
Moreover, for every $m \in \mathbb{N}$ it holds

$$\sup_{t \in (0,T]} \left(t^m \|H_{E_t^h}\|^2_{H^m(\partial E_t^h)} \right) \leq C_m.$$

The proof is inspired by Ishii-Lions (1990), also uses the two-point function by Huisken.

Corollary (Julin-Niinikoski 2022)

Assume that $E_0 \subset \mathbb{R}^{n+1}$ satisfies interior and exterior ball condition. Then the flat flow agrees with the classical solution of the VMCF starting from E_0 as long as the latter exists.

Related result: Distributional solution agrees with the classical solution, solutions obtain via phase-field approximation (Laux 2022).

Partial Regularity:
The advantage of having a Brakke flow is that it implies partial regularity. Can we have it for the flat flow? The following holds for the MCF

$$V_t = -H_{E_t}.$$

Theorem (in preparation....Arya-Jeon-Julin)

Let (E_t) be a flat flow solution (E_t^h the approximation) to MCF and assume

(1) $\partial E_t \cap B_2 \subset \{x \in \mathbb{R}^{n+1} : |x_{n+1}| < \varepsilon\}$ *for all* $t \in [t_0 - 1, t_0]$

(2) $\lim_{h\to 0} P(E^h_{t_0-1}; B_1) \leq (1+\varepsilon)|B_1^n|$.

Then there is $\eta \in (0,1)$ *such that* $\partial E_t \cap B_\eta$ *is smooth for all* $t \in (t_0 - \eta^2, t_0]$.

Asymptotic

Since the flat flow is defined for all times, we may study the asymptotic limit of

$$V_t = -H_{E_t} + \bar{H}_{E_t}$$

when $t \to \infty$. Heuristically if $E_t \to E_\infty$ then

$$H_{E_\infty} = \bar{H}_{E_\infty} = \text{constant}$$

and by Alexandrov theorem E_∞ is union of balls.

As an example, study the gradient flow of smooth uniformly convex function $F : \mathbb{R}^n \to \mathbb{R}$, minimum point at 0 with $F(0) = 0$.

$$\frac{d}{dt}x(t) = -\nabla F(x(t)), \qquad x(0) = x_0$$

Uniform convexity implies the *Lojasiewicz inequality*

$$cF(x) \leq |\nabla F(x)|^2$$

Therefore

$$\frac{d}{dt}F(x(t)) = -|\nabla F(x(t))|^2 \leq -cF(x(t)).$$

which implies the exponential convergence

$$F(x(t)) \leq F(x_0)e^{-c\,t}.$$

Analogue in the VMCF: $V_t = -H_{E_t} + \bar{H}_{E_t}$.

$$\frac{d}{dt}P(E_t) = \int_{\partial E_t} H_{E_t} V_t \, d\mathcal{H}^n = -\int_{\partial E_t} (H_{E_t} - \bar{H}_{E_t})^2 \, d\mathcal{H}^n$$

Integrate over (T, ∞)

$$\int_T^\infty \|H_{E_t} - \bar{H}_{E_t}\|_{L^2}^2 \, dt = P(E_T) - P(E_\infty)$$

Assume that we know that $E_\infty = B_1$ and we have the *Lojasiewicz inequality*, for all $|E| = |B_1|$

$$P(E) - P(B_1) \leq C\|H_E - \bar{H}_E\|_{L^2}^2$$

Then

$$F(T) := \int_T^\infty \|H_{E_t} - \bar{H}_{E_t}\|_{L^2}^2 \, dt = P(E_T) - P(B_1)$$
$$\leq C\|H_{E_T} - \bar{H}_{E_T}\|_{L^2}^2 = -CF'(T).$$

The key inequality is, $|E| = |B_1|$ with $P(E) \leq C_0$

$$\min_{d \in \mathbb{N}} |P(E) - P_d| \leq C \|H_E - \bar{H}_E\|_{L^2}^2 \tag{1}$$

The inequality (1) is related to so called *quantitative Alexandrov inequality*, Ciraolo-Maggi 2017.... We know (1) in 2D (Julin-Morini-Ponsiglione-Spadaro 2022, Kim-Kwon 2024). In $\mathbb{R}^{n+1}$ with $n \geq 3$ we know nothing. In 3D we have...

Theorem (Julin-Niinikoski (2020))

For $E \subset \mathbb{R}^3$ with $|E| = |B_1|$ and $P(E) \leq C_0$ it holds

$$\min_{d \in \mathbb{N}} |P(E) - P_d| \leq C \|H_E - \bar{H}_E\|_{L^2}^q$$

and there is a union of disjoint spheres $\partial F = \bigcup_{i=1}^d \partial B_r(x_i)$

$$d_{\mathcal{H}}(\partial E, \partial F) \leq C \|H_E - \bar{H}_E\|_{L^2}^q.$$

The non-sharp Alexandrov implies qualitative convergence of the VMCF

Theorem (Julin-Niinikoski (2020))

Assume $(E_t)_{t\geq 0}$ with $|E_t| = |B_1|$ is a flat flow solution to the VMCF

$$V_t = -H_{E_t} + \bar{H}_{E_t}$$

starting from an open and bounded set of finite perimeter $E_0 \subset \mathbb{R}^3$. Then there is d such that for every $\varepsilon > 0$ there is T_ε such that ∂E_t is close to a union d many spheres $\partial F(t)$ in Hausdorff

$$d_{\mathcal{H}}(\partial E_t, \partial F(t)) \leq \varepsilon \qquad \text{for all } t \geq T_\varepsilon.$$

Note that $\partial F(t)$ may depend on time. If $d = 1$ we may remove this.

Theorem (Julin-Morini-Oronzio-Spadaro (2024))

For $E \subset \mathbb{R}^3$ *with* $|E| = |B_1|$ *and* $P(E) \le 4\pi \sqrt[3]{2} - \delta_0$ *it holds*

$$|P(E) - P(B_1)| \le C\|H_E - \bar{H}_E\|_{L^2}^2.$$

Theorem (Julin-Morini-Oronzio-Spadaro (2024))

Assume $(E_t)_{t \ge 0}$ *with* $|E_t| = |B_1|$ *is a flat flow solution to the VMCF*

$$V_t = -H_{E_t} + \bar{H}_{E_t}$$

starting from an open and bounded $E_0 \subset \mathbb{R}^3$ *with* $P(E_0) \le 4\pi \sqrt[3]{2} - \delta_0$. *Then there is* $x_0 \in \mathbb{R}^3$ *such that*

$$d_{\mathcal{H}}(\partial E_t, \partial B_1(x_0)) \le Ce^{-ct}.$$

The exponential convergence also holds if in the qualitative convergence E_t *converges to union of balls with positive distance!*

A few words about the proof of: $E \subset \mathbb{R}^3$ with $|E| = |B_1|$ and $P(E) \leq 4\pi \sqrt[3]{2} - \delta_0$

$$P(E) - P(B_1) \leq C\|H_E - \bar{H}_E\|_{L^2}^2 = C \int_{\partial E} H_E^2 - \bar{H}_E^2 \, d\mathcal{H}^2. \qquad (2)$$

In $\mathbb{R}^3$ the Willmore energy $\int_{\partial E} H_E^2 \, d\mathcal{H}^2$ is conformal invariant and it holds

$$\int_{\partial E} H_E^2 \, d\mathcal{H}^2 \geq \int_{\partial B_1} H_{B_1}^2 \, d\mathcal{H}^2 = 16\pi$$

The inequality (2) is stronger than the quantitative Willmore inequality by Röger-Schätzle (2012)

$$P(E) - P(B_1) \leq C \left(\int_{\partial E} H_E^2 \, d\mathcal{H}^2 - 16\pi \right).$$

We show that when $\varepsilon_0 > 0$ is small the unique minimizer of the Canham-Helfrich type energy

$$J(E) := \int_{\partial E} (H_E - \bar{H}_E)^2 \, d\mathcal{H}^2 - \varepsilon_0 P(E) \tag{3}$$

under the volume constraint $|E| = |B_1|$ and $P(E) \leq 4\pi \sqrt[3]{2} - \delta_0$ is the ball.

- Using apriori estimates from Julin-Niinikoski (2020) we write the energy $J(E)$ in terms *weak immersions* of the sphere $\mathbb{S}^2 = \partial B_1$.
- Using results by Mondino, Riviere, Scharrer we obtain that the minimizer E of (3) exists and is smooth.

The Euler-Lagrange equation reads as

$$-\Delta_{\partial E} H_E = |B_E|^2 (H_E - \bar{H}_E) + \text{other terms}$$

Usually the PDE of type

$$-\Delta u = a(x) u^3 + \text{l.o.t}$$

does not imply uniform regularity estimates. However, if you know that u is small, then the solution is uniformly regular. Here the situation is similar and we are able to prove that

$$\|\nabla^2 H_E\|_{L^2(\partial E)} \leq C.$$

This gives uniform $C^{2,\alpha}$-estimates which is enough to prove the statement.

Thank you!