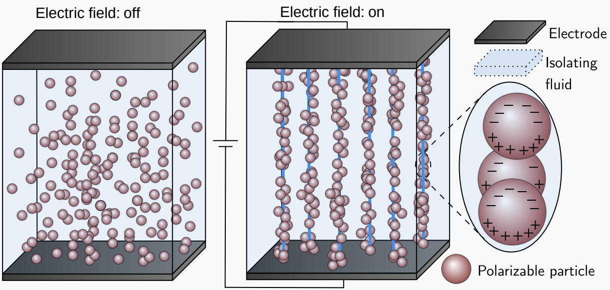
Numerical Methods for Smart Fluids

Error analysis for a finite element approximation of a simplified model

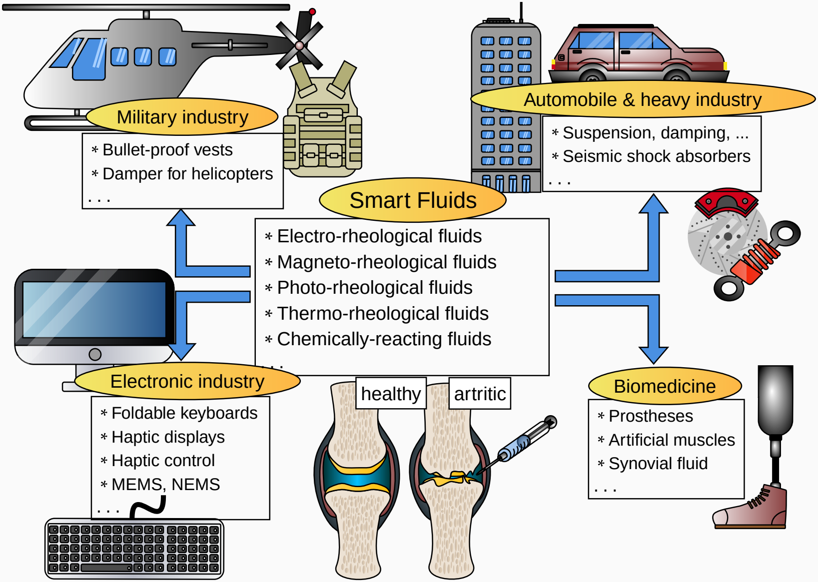
Luigi C. Berselli, Alex Kaltenbach

September 24, 2024

Modelling, PDE analysis and computational mathematics in materials science



Areas of application



Behera, A., *Advanced Materials: An Introduction to Modern Materials Science*, **Springer Cham**, 2022.
https://doi.org/10.1007/978-3-030-80359-9

The steady $p(x)$-Navier–Stokes equations

- **Steady $p(x)$-Navier–Stokes equations:** $(\mathbf{D}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top))$

  Seek *velocity vector field* $\mathbf{v}\colon \overline{\Omega} \to \mathbb{R}^d$, $d \in \{2,3\}$, and *pressure* $q\colon \Omega \to \mathbb{R}$ such that

$$\left.\begin{aligned} -\operatorname{div}\mathbf{S}(\cdot,\mathbf{D}\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla q &= \mathbf{f} && \text{in } \Omega\,,\\ \operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega\,,\\ \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega\,, \end{aligned}\right\} \qquad (p(x)\text{-NSE})$$

  where for $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, with $p^- := \min_{x\in\overline{\Omega}} p(x) > \frac{3d}{d+2}$ and $\delta > 0$, we have that

$$\mathbf{S}(\cdot,\mathbf{D}\mathbf{v}) \coloneqq (\delta + |\mathbf{D}\mathbf{v}|)^{p(\cdot)-2}\mathbf{D}\mathbf{v}\,.$$

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- **Related contributions:**
  - → **Error analyses:**
    - [4, Breit, Diening, Schwarzacher, '15] ($p(x)$-Laplace equation);
    - [2, Berselli, Breit, Diening, '16] ($p(x)$-Stokes equations);
  - → **Convergence analyses:**
    - [5, Del Pezzo, Lombardi, Martínez, '12] ($p(x)$-Laplace equation);
    - [7, Ko, Pustějovská, Süli, '18], [8, Ko, Süli, '19] ($p(x)$-Navier–Stokes equations).

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- **Functional analytical framework:**

  → **Energy estimate:**

$$|\mathbf{Dv}| \in L^{p(\cdot)}(\Omega) \coloneqq \left\{z \in L^0(\Omega) \mid |z|^{p(\cdot)} \in L^1(\Omega)\right\}.$$

  → **Energy spaces:**

$$\begin{aligned} \mathbf{v} &\in (W^{1,p(\cdot)}_0(\Omega))^d \coloneqq \left\{\mathbf{z} \in (W^{1,1}_0(\Omega))^d \mid \nabla\mathbf{z} \in (L^{p(\cdot)}(\Omega))^{d\times d}\right\},\\ q &\in L^{p'(\cdot)}(\Omega)\,, \end{aligned}$$

  where $p' \in C^{0,\alpha}(\overline{\Omega})$ is defined by $p'(x) := \frac{p(x)}{p(x)-1}$ for all $x \in \Omega$.

Continuous and discrete formulation

- **Continuous problem:** Seek $(\mathbf{v}, q)^\top \in (W_0^{1,p(\cdot)}(\Omega))^d \times (L^{p'(\cdot)}(\Omega)/\mathbb{R})$ such that

$$
\begin{aligned}
(\mathbf{S}(\cdot, \mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{z})_\Omega - (\mathbf{v} \otimes \mathbf{v}, \mathbf{D}\mathbf{z})_\Omega - (q, \operatorname{div} \mathbf{z})_\Omega &= (\mathbf{f}, \mathbf{z})_\Omega \,, \\
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- **Discretized extra-stress tensor:**

$$\mathbf{S} \quad \leftrightarrow \quad \mathbf{S}_h$$

  where $\mathbf{S}_h \colon \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, for every $T \in \mathcal{T}_h$, is defined by

$$\mathbf{S}_h(x, \cdot) \coloneqq \mathbf{S}(\xi_T, \cdot) \quad \text{for all } x \in T \,,$$

  where $\xi_T \in T$ is an arbitrary quadrature point.

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- **Discrete problem:** Seek $(\mathbf{v}_h, q_h)^\top \in V_h \times (Q_h/\mathbb{R})$ such that

$$\begin{aligned}(\mathbf{S}_h(\cdot, \mathbf{D}\mathbf{v}_h), \mathbf{D}\mathbf{z}_h)_\Omega - \tfrac{1}{2}(\mathbf{v}_h \otimes \mathbf{v}_h, \mathbf{D}\mathbf{z}_h)_\Omega + \tfrac{1}{2}(\mathbf{z}_h \otimes \mathbf{v}_h, \mathbf{D}\mathbf{v}_h)_\Omega - (q_h, \operatorname{div} \mathbf{z}_h)_\Omega &= (\mathbf{f}, \mathbf{z}_h)_\Omega \,, \\ (\operatorname{div} \mathbf{v}_h, z_h)_\Omega &= 0 \,,\end{aligned}$$

  for all $(\mathbf{z}_h, z_h)^\top \in V_h \times Q_h$, where (V_h, Q_h) is a discretely inf-sup stable FE couple (*e.g.*, MINI, P2P0, Taylor–Hood...).

'Natural' (fractional) regularity assumptions

- **'Natural' regularity on the velocity:** $(\mathbf{F}(\cdot, \mathbf{Dv}) := (\delta + |\mathbf{Dv}|)^{\frac{p(\cdot)-2}{2}}\mathbf{Dv})$

  → **'Full' regularity:** (*cf.* [1, Acerbi, Mingone, '02])

$$p \in C^{0,1}(\overline{\Omega}) \qquad \Rightarrow \qquad \mathbf{F}(\cdot, \mathbf{Dv}) \in (W^{1,2}_{\mathrm{loc}}(\Omega))^{d\times d}\,.$$

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where the *fractional variable Hajłasz–Sobolev space* is defined by

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where the *fractional variable Hajlasz–Sobolev space* is defined by

$$H^{\alpha,p'(\cdot)}(\Omega) := \{u \in L^{p'(\cdot)}(\Omega) \mid G_\alpha(u) \neq \emptyset\}\,,$$

$$G_\alpha(u) := \{g \in L^{p'(\cdot)}(\Omega) \mid |u(x)-u(y)| \le (g(x)+g(y))|x-y|^\alpha \text{ for a.e. } x,y \in \Omega\}\,.$$

A priori error estimate for the velocity

Theorem (*a priori* error estimate for the velocity)

If $p \in C^{0,\alpha}(\overline{\Omega})$, $\mathbf{F}(\cdot, \mathbf{Dv}) \in (N^{\alpha,2}(\Omega))^{d \times d}$, $q \in H^{\alpha,p'(\cdot)}(\Omega)$, *and* $\|\nabla \mathbf{v}\|_{2 \wedge p(\cdot),\Omega} \ll 1$, *then*

$$\|\mathbf{F}(\cdot, \mathbf{Dv}_h) - \mathbf{F}(\cdot, \mathbf{Dv})\|_{2,\Omega}^2 \lesssim h^{\min\{2,(p^+)'\}\alpha}\,.$$

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- **Proof.**
  - **1. Step**: *(Best-approximation type estimate)*

$$\begin{aligned}\|\mathbf{F}(\cdot,\mathbf{Dv}) - \mathbf{F}(\cdot,\mathbf{Dv}_h)\|_{2,\Omega}^2 \lesssim\ & \|(\varphi_{|\mathbf{Dv}|})^*(|\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_h(\cdot,\mathbf{Dv})|)\|_{1,\Omega} \\ & + \inf_{\mathbf{z}_h\in V_h}\big\{\|\mathbf{F}(\cdot,\mathbf{Dv}) - \mathbf{F}(\cdot,\mathbf{Dz}_h)\|_{2,\Omega}^2\big\} \\ & + \inf_{z_h\in Q_h}\big\{\|(\varphi_{|\mathbf{Dv}|})^*(|q - z_h|)\|_{1,\Omega}\big\}\,,\end{aligned}$$

where $(\varphi_{|\mathbf{Dv}|})^*(\cdot,t) \sim ((\delta + |\mathbf{Dv}|)^{p(\cdot)-1} + t)^{p'(\cdot)-2}t^2$ uniformly for all $t \geq 0$.

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  - → **2. Step**: *(Oszillation/Expressivity estimates)*

$$\begin{aligned}\|(\varphi_{|\mathbf{Dv}|})^*(|\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_h(\cdot,\mathbf{Dv})|)\|_{1,\Omega} &\lesssim h^{2\alpha}\,, \\ \inf_{\mathbf{z}_h\in V_h}\big\{\|\mathbf{F}(\cdot,\mathbf{Dv}) - \mathbf{F}(\cdot,\mathbf{Dz}_h)\|_{2,\Omega}^2\big\} &\lesssim h^{2\alpha}\,; \\ \inf_{z_h\in Q_h}\big\{\|(\varphi_{|\mathbf{Dv}|})^*(|q - z_h|)\|_{1,\Omega}\big\} &\lesssim h^{\min\{2,(p^+)'\}\alpha}\,.\end{aligned}$$

∎

Numerical experiments: velocity error

- **Experimental setup:** Let $d = 2$, $\Omega = (0,1)^2$, $\delta = 1\mathrm{e}{-5}$, $\alpha \in (0,1]$,

$$p \coloneqq \Big(1 - \frac{|\cdot|^{\alpha}}{2^{\alpha/2}}\Big)\, p^+ + \frac{|\cdot|^{\alpha}}{2^{\alpha/2}}\, p^- \,, \qquad \mathbf{v} \coloneqq |\cdot|^{2\frac{\alpha-1}{p}+\delta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\cdot)\,, \qquad q \coloneqq |\cdot|^{\alpha - \frac{2}{p'} + \delta}\,.$$

i \ p^-	1.5	1.75	2.0	2.25	2.5	2.75
α	1.0					
4	0.750	0.719	0.695	0.676	0.659	0.645
5	0.801	0.757	0.725	0.701	0.681	0.665
6	0.824	0.774	0.739	0.713	0.692	0.674
7	0.833	0.782	0.746	0.718	0.696	0.678
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7	0.486	0.438	0.393	0.365	0.348	0.338
8	0.476	0.430	0.390	0.365	0.350	0.339
9	0.470	0.425	0.388	0.365	0.350	0.341
theory	0.417	0.393	0.375	0.361	0.350	0.341

Table: Experimental order of convergence (MINI element): $\texttt{EOC}_i\Big(\|\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{D}\mathbf{v}_{h_i})\|_{2,\Omega}\Big)$, $i = 4,\dots,9$.

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Thank You!

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