Variational aspects of fluid-structure interaction

based on joint work[1] with B. Benešová & S.Schwarzacher

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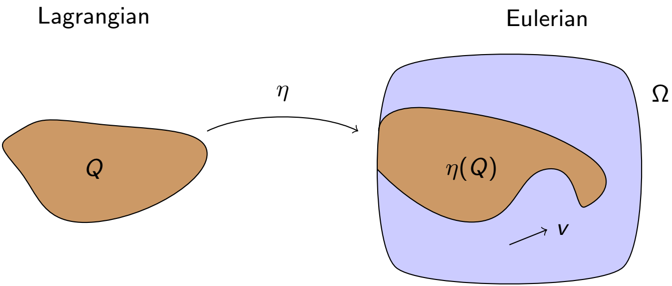
Fluid structure interaction



System of equations

$$\begin{cases} \rho_s \partial_t^2 \eta + DE(\eta) + D_2 R(\eta, \partial_t \eta) = f_s & \text{in } Q \\ \rho_f (\partial_t + v \cdot \nabla) v - \nu \Delta v + \nabla p = f_f & \text{in } \Omega(t) := \Omega \setminus \eta(t, Q) \\ \nabla \cdot v = 0 & \text{in } \Omega(t) \\ v \circ \eta = \partial_t \eta & \text{in } \partial Q \end{cases}$$

+ equivalence of forces at the interface + boundary & initial data.

Sources of nonlinearities & nonconvexities

$$\begin{cases} \rho_s \partial_t^2 \eta + DE(\eta) + D_2 R(\eta, \partial_t \eta) = f_s & \text{in } Q \\ \rho_f (\partial_t + v \cdot \nabla) v - \nu \Delta v + \nabla p = f_f & \text{in } \Omega(t) \\ v \circ \eta = \partial_t \eta & \text{in } \partial Q \end{cases}$$

In the equation

- Transport terms $(\partial_t + v \cdot \nabla) v$
- Large strain elasticity DE (in general)
- Injectivity constraints $\det \nabla \eta > 0$

In boundary/coupling conditions

- Change between Lagrangian and Eulerian reference $v \circ \eta = \partial_t \eta$

In the domain itself

- $\Omega(t) := \Omega \setminus \eta(Q, t)$, $v : \Omega(t) \to \mathbb{R}^n$

The three regimes of continuum mechanics

Static ($\neq$ stationary)

Goal: Find a stable stationary point of the potential energy

$$DE(\eta) = f_s, \qquad\qquad v = 0$$

- Clear formulation as **minimization problem**
- Extremly well studied, mostly from **variational** point of view

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Quasistatic

Goal: Evolve slowly, trading energy against dissipation

$$D_2 R(\eta, \partial_t \eta) + DE(\eta) = f_s, \qquad -\nu \Delta v + \nabla p = f_f$$

- **Gradient flow structure**
- Amenable to variational methods (**Minimizing movements**)

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Dynamic (inertial)

Goal: Evolve at higher speeds, where conservation of momentum becomes significant

$$\rho_s \partial_{tt} \eta + D_2R(\eta, \partial_t \eta) + DE(\eta) = f_s, \qquad \rho_f (\partial_t + v \cdot \nabla) v - \nu \Delta v + \nabla p = f_f$$

- Variational structure sometimes present, but not in form of minimizers
- Classicaly the **realm of PDE methods**, which **do not cope well with non-convexity**

Time-delayed approach [Benešová, K., Schwarzacher '24]

Time-delayed equation

Why not solve

$$\rho\frac{\partial_t\eta - \partial_t\eta(\cdot - h)}{h} + DE(\eta) + D_2R(\eta,\partial_t\eta) = \text{forces} \qquad \text{(TD)}$$

instead?

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instead?

General plan

- On $[0, h]$: treat $\rho \frac{\partial_t \eta(\cdot - h)}{h}$ as fixed data and $\rho \frac{\partial_t \eta}{h} = D_{\partial_t \eta} \int_Q \frac{\rho}{h} |\partial_t \eta|^2 \, dx$

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- Testing (TD) with $\partial_t\eta$ (or using De Giorgi estimate) gives

$$\fint_{t-h}^{t}\frac{\rho}{2}\|\partial_t\eta\|^2\,dt + E(\eta(t)) + \int_0^t R + R^*\,dt \le E(\eta_0) + \fint_{0-h}^{0}\frac{\rho}{2}\|\partial_t\eta\|^2\,dt + W_{\text{forces}}$$

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  $\Rightarrow$ allows to iterate on $[h, 2h]$, $[2h, 3h]$, ...
- Energy estimate telescopes and is uniform in h
  $\Rightarrow$ limit $h \to 0$ is possible on the level of the PDE

An existence result

Theorem [Benešová, K., Schwarzacher '24]

Assume that E and R satisfy some conditions and Q and Ω are regular enough. Then there exists a weak solution to the full bulk FSI-problem up until the first (self-)contact of the solid. Additionally the solution obeys an energy inequality.

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[1]Terms and conditions apply:

- Energy $E : W^{2,q} \to \mathbb{R}$, $q > n$ is coercive, weakly lsc., bounded from below, DE exists, has some sort of continuity as well as a Minty-type property, E finite implies $\det \nabla\eta > 0$ uniformly
- Dissipation R is 2-homogeneous, weakly lsc., has a Korn inequality for bounded energies, D_2R exists and has some sort of continuity
- Q and Ω are Lipschitz and piecewise $C^{1,\alpha}$.

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Covers a large class of physical reasonable materials. In particular E and space of deformations can be highly non-convex.

Closing remarks

Some other results using the method

- **Existence for compressible, bulk FSI** [Breit, K., Schwarzacher '24]
- Poroelasticity [Benešová, K., Schwarzacher '23]
- Thin solids [K., Schwarzacher, Sperone '23]
- FSI past contact [K.,Muha,Trifunović '24]
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Other aspects

- The method is connected to an **energetical modelling procedure** (Energy+Dissipation+kinematics $\Rightarrow$ weak/measure valued-solution)
- The method is well suited to coupling different problems (**Fluid-structure interaction**, multiphysics)
- Many possible applications in progress/still unexplored

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Thank you for your attention.