Stability of steady states to generalized Navier-Stokes-Fourier system

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Equations:

$$\partial_t \mathrm{v} + \operatorname{div}(\mathrm{v} \otimes \mathrm{v}) - \operatorname{div} \mathrm{S} + \nabla \pi = 0, \quad \operatorname{div} \mathrm{v} = 0$$
$$\partial_t \theta + \operatorname{div}(\theta \mathrm{v}) + \operatorname{div} \mathrm{q} = \mathrm{S} : \mathrm{Dv}$$

Unknowns: $\mathrm{v} : Q \to \mathbb{R}^3$, $\theta, \pi : Q \to \mathbb{R}$, where $Q = (0, +\infty) \times \Omega$ and $\Omega \subset \mathbb{R}^3$ open, bounded with Lipschitz boundary, $I = (0, +\infty)$

Constitutive relations: $\mathrm{S} = \nu(\theta)(1 + |\mathrm{Dv}|^2)^{\frac{p-2}{2}} \mathrm{Dv}$ for some $p > 1$, $\mathrm{q} = -\kappa(\theta)\nabla\theta$ for a κ, ν positive, bounded and bounded away from 0.

Boundary conditions: $\mathrm{v} = 0$, $\theta = \theta_b$ on $I \times \partial\Omega$, assume $0 < \underline{\theta} \leq \theta_b$
Initial conditions: $\mathrm{v} = \mathrm{v}_0$, $\theta = \theta_0$ in $\{0\} \times \Omega$, assume $0 < \underline{\theta} \leq \theta_0$

Stationary solution/steady state:
$\hat{\mathrm{v}} = 0$, $\operatorname{div}(\mathrm{q}(\hat{\theta}, \nabla\hat{\theta}) = 0$ in Ω, $\hat{\theta} = \theta_b$ on $\partial\Omega$,
$\hat{\theta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $0 < \underline{\theta} \leq \hat{\theta} \leq \overline{\theta}$.

Choice of Lyapunov functional

[Clausius, 1865] - The energy of the world is constant. The entropy of the world strives to maximum.
[Bulíček et al., 2019] - based on second law of thermodynamic
[Dostalík et al., 2021] - suitable choice of temperature scale

Candidate for Lyapunov functional: $L = \mathcal{E}(\mathrm{v}, \theta, 0, \hat{\theta}) - \mathcal{S}(\mathrm{v}, \theta, 0, \hat{\theta})$
In our situation (with $\mathrm{q} = -\nabla\theta$ and $\alpha \in (0,1)$):
Try 1:

$$L_1(\mathrm{v}, \theta) = \int_\Omega \frac{1}{2}|\mathrm{v}|^2 + \theta - \hat{\theta} - \hat{\theta}\lg\left(\frac{\theta}{\hat{\theta}}\right).$$

Try 2:

$$L_\alpha(\mathrm{v}, \theta) = \int_\Omega \frac{1}{2}|\mathrm{v}|^2 + \theta - \hat{\theta} - \frac{\hat{\theta}}{1-\alpha}\left(\left(\frac{\theta}{\hat{\theta}}\right)^{1-\alpha} - 1\right).$$

...convergence of smooth solutions to steady state, no rate of convergence

relative entropy/energy functional

Relative entropy inequality $\alpha \in (0,1]$

$$\partial_t L_\alpha(\mathsf{v},\theta) + \int_\Omega \alpha\hat{\theta}\Big(\frac{\theta}{\hat{\theta}}\Big)^{-1-\alpha}\Big|\nabla\Big(\frac{\theta}{\hat{\theta}}\Big)\Big|^2 + \int_\Omega \Big(\frac{\theta}{\hat{\theta}}\Big)^{-\alpha}\mathsf{S} : \mathsf{D}\mathsf{v} \leq \int_\Omega \mathsf{v}\nabla\theta\Big(\frac{\theta}{\hat{\theta}}\Big)^{-\alpha}.$$

Tasks:

estimates integrals on the LHS from below with $\mu L_\alpha(\mathsf{v},\theta)$, $\mu > 0$
estimate the remainder of the transport term on the RHS

A-priori estimates:

$$\|\mathsf{v}\|_{L^\infty(0,+\infty;L^2(\Omega))}, \|\theta\|_{L^\infty(0,+\infty;L^1(\Omega))}, \int_0^{+\infty}\int_\Omega \mathsf{S} : \mathsf{D}\mathsf{v}$$

by data.

Relative entropy inequality with added kinetic energy

$$L_{\alpha,\beta}(\mathrm{v},\theta) = \beta \|\mathrm{v}\|^2_{L^2(\Omega)} + L_\alpha(\mathrm{v},\theta)$$

for $\alpha \in (0,1]$, $\beta \geq 0$

There is $\mu > 0$, $\beta > 0$ that depend only on size of data $\hat{\theta}, \theta_0, \mathrm{v}_0$ such that

$$\partial_t L_{\alpha,\beta} + \mu L_{\alpha,\beta} \leq 0.$$

$\rightarrow$ Exponential decay of the functional $L_{\alpha,\beta}$ to zero.

Theorem

$p > 6/5$

$\implies$ there exists $\mu > 0$ s.t for almost every $\tau > 0$

$$\|\mathrm{v}(\tau)\|_2^2 \leq e^{-\mu\tau} C(\|\mathrm{v}(0)\|_2),$$

$p \geq 8/5$, $\alpha \in (\max(1/2, 2 - 5p/6), 2/3]$ and $R > 0$

$\implies$ there exist $\mu, \beta > 0$ such that for almost every $\tau \geq 0$

$$\|\mathrm{v}_0\|_2^2 + \|\theta_0\|_1 \leq R \implies L_{\alpha,\beta}(\mathrm{v}(\tau), \theta(\tau)) \leq e^{-\mu\tau} L_{\alpha,\beta}(\mathrm{v}(0), \theta(0))$$

provided (v, θ) is a solution satisfying kinetic energy inequality, relative entropy inequality, and uniform estimates in time of

$$\|\mathrm{v}\|_{L^\infty(0,+\infty;L^2(\Omega))}, \|\theta\|_{L^\infty(0,+\infty;L^1(\Omega))}, \int_0^{+\infty} \int_\Omega \mathrm{S} : \mathrm{Dv}$$

by data.

How to obtain solutions satisfying relative entropy inequality?

1. Derive the relative entropy inequality from definition of the weak solution
2. Add some physical (in)equality to the notion of the weak solution in such a way that the solution satisfies relative entropy inequality
    1. relative entropy inequality
    2. entropy equality
    3. modified total energy inequality

Add the relative entropy inequality to the notion of the solution

- It is possible in our situation.
- Done for compressible NSF system with the forcing θe_3 in the balance of linear momentum, see [Feireisl et al., 2024]
- no exponential convergence to steady state

Relative entropy inequality $\alpha \in (0,1]$

$$\partial_t L_\alpha(\mathrm{v},\theta) + \int_\Omega \alpha\hat{\theta}\left(\frac{\theta}{\hat{\theta}}\right)^{-1-\alpha}\left|\nabla\left(\frac{\theta}{\hat{\theta}}\right)\right|^2 + \int_\Omega \left(\frac{\theta}{\hat{\theta}}\right)^{-\alpha}\mathrm{S} : \mathrm{Dv} \le \int_\Omega \mathrm{v}\nabla\theta\left(\frac{\theta}{\hat{\theta}}\right)^{-\alpha}.$$

Add entropy equality to the notion of the weak solution

- Existence of such solutions proved in [Abbatiello et al., 2022] for $p \geq 11/5$ if $\Omega \subset \mathbb{R}^3$.
- Stability of such solution proved in [Abbatiello et al., 2024].
- 2D situation studied in [Wintrová, 2024].

Entropy

$$\eta = \lg(\theta), \qquad \partial_t \eta + \operatorname{div}(\eta \mathrm{v}) - \operatorname{div}(\kappa(\theta)\nabla\eta) = \frac{\mathrm{S} : \mathrm{Dv}}{\theta} + \kappa(\theta)|\nabla\eta|^2$$

- Work in progress: generalization to $\kappa(\theta) = \theta^\beta$, $\beta \in \mathbb{R}$ [Hajduk, Wroblewska, K.]—for $\beta \geq 0$, need for bounded entropies $\eta_\gamma = (\underline{\theta}^{1-\gamma} - \theta^{1-\gamma})/(\gamma - 1)$ for $\gamma > 1$

Add modified total energy equality to the notion of the weak solution

- Stability for such solution if $p \geq 8/5$.
- Weak solution+kinetic energy inequality, entropy inequality, modified total energy inequality

Modified total energy inequality

$$\partial_t \int_\Omega \left[\frac{|\mathrm{v}|^2}{2} + \theta - B(\theta, \hat{\theta})\,\varphi \right] + \int_\Omega \mathrm{v} \cdot \nabla\varphi\, B(\theta, \hat{\theta}) + \int_\Omega \mathrm{v} \cdot \nabla\hat{\theta}\, \partial_2 B(\theta, \hat{\theta})\,\varphi$$
$$- \int_\Omega \kappa(\theta)\nabla\theta \cdot \nabla\varphi\, b(\theta, \hat{\theta}) - \int_\Omega \kappa(\theta)\nabla\theta \cdot \nabla\hat{\theta}\, \partial_2 b(\theta, \hat{\theta})\,\varphi \leq 0$$

for all $\varphi \in C^\infty(\overline{\Omega})$ s.t. $\varphi = 1$ on $\partial\Omega$.
b is some function s.t. $b(s, s) = 1$ and $\partial_1 B = b$.
Obtained by testing balance of linear momentum with v and the heat equation with $(1 - b(\theta, \hat{\theta})\varphi)$.

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