On time-periodic solutions to an interaction problem between compressible viscous fluids and viscoelastic beams

Ondřej Kreml

Institute of Mathematics, Czech Academy of Sciences,
Nečas Center for Mathematical Modeling

Joint work with Václav Mácha, Šárka Nečasová and Srđan Trifunović

MPDE 2024, Prague, September 23, 2024

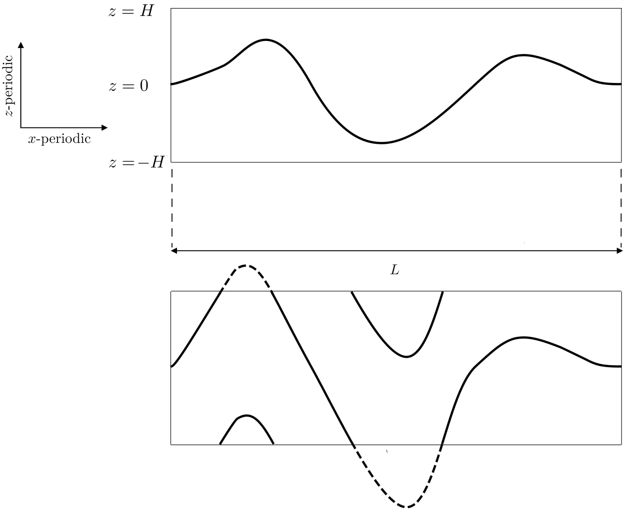
Previous work

- **Time periodic weak solutions for compressible fluids**:
  - barotropic case: Feireisl, Nečasová, Petzeltová, Straškraba (1999)
  - full system with temperature: Feireisl, Mucha, Novotný, Pokorný (2012)
- **Weak solutions for interaction of compressible fluids with elastic shells**:
  - Koiter shell: Breit, Schwarzacher (2018)
  - Thermoelastic shell: Trifunović, Wang (2023)
- **Time periodic interaction problems**:
  - incompressible 2D fluid, strong solution: Casanova (2019)
  - incompressible 3D fluid, weak solution: Mîndrilă, Schwarzacher (2022, 2023)

We consider a rectangular domain $\Omega \subset \mathbb{R}^2$ filled with a compressible viscous fluid and containing a viscoelastic beam clamped on the sides of the domain.
In order to avoid the beam from touching the boundary, we make a significant technical simplification: periodicity in the z-direction.
Notation:

$\Gamma = (0, L)$	$\Gamma_T = (0, T) \times \Gamma$
$\Omega = (0, L) \times (-H, H)$	domain filled with the fluid
$Q_T = (0, T) \times \Omega$	space-time fluid cylinder
(t, x, z)	time, horizontal and vertical variables
$\eta(t, x)$	vertical displacement of the beam
$\hat{\eta}(t, x)$	$= \eta(t, x) - 2nH \in [-H, H)$
$\Gamma^\eta(t) = \{(x, \eta(t, x)) : x \in \Gamma\}$	position of the beam
ρ, u	density and velocity of the fluid

z-periodic version of the beam



Governing equations

The viscoelastic beam equation on Γ_T:

$$\eta_{tt} + \eta_{xxxx} - \eta_{txx} = -S^\eta \mathrm{f}_{fl} \cdot \mathrm{e}_2 + f.$$

$S^\eta = \sqrt{1 + |\eta_x|^2}$ ... Jacobian of the transformation from Eulerian to Lagrangian coordinates,
f ... time-periodic force
The compressible Navier-Stokes equations on Q_T:

$$\begin{aligned} \partial_t(\rho \mathrm{u}) + \nabla \cdot (\rho \mathrm{u} \otimes \mathrm{u}) &= -\nabla p(\rho) + \nabla \cdot \mathbb{S}(\nabla \mathrm{u}) + \rho \mathrm{F}, \\ \partial_t \rho + \nabla \cdot (\rho \mathrm{u}) &= 0, \end{aligned}$$

where for simplicity $p(\rho) = \rho^\gamma$ and

$$\mathbb{S}(\nabla \mathrm{u}) := \mu\big(\nabla \mathrm{u} + \nabla^\tau \mathrm{u} - \nabla \cdot \mathrm{u}\mathbb{I}\big) + \zeta \nabla \cdot \mathrm{u}\mathbb{I}, \quad \mu, \zeta > 0.$$

F is the time-periodic force acting onto the fluid.

The fluid-structure coupling (kinematic and dynamic, resp.) on Γ_T:

$$\begin{aligned} \eta_t(t,x)\mathrm{e}_2 &= \mathrm{u}(t,x,\hat{\eta}(t,x)), \\ \mathrm{f}_{fl}(t,x) &= \big[\big[(-p(\rho)\mathbb{I} + \mathbb{S}(\nabla \mathrm{u}))\big]\big](t,x,\hat{\eta}(t,x))\ \nu^\eta(t,x), \end{aligned}$$

where $\nu^\eta = \frac{(-\eta_x,1)}{\sqrt{1+|\eta_x|^2}}$ denotes normal vector on Γ^η facing upwards and

$$[[A]](\cdot,z) := \lim_{\varepsilon\to 0^+} \big(A(\cdot,z-\varepsilon) - A(\cdot,z+\varepsilon)\big).$$

represents the jump in the vertical direction.

Boundary conditions

The beam boundary conditions (clamped):

$$\eta \text{ is periodic in } x \text{ and } \eta(t,x) = 0, \quad (t,x) \in (0,T) \times \{0,L\}.$$

Fluid spatial periodicity:

$$\rho, \mathrm{u} \text{ are periodic in } x \text{ and } z \text{ directions.}$$

Time periodicity:

$$\rho, \mathrm{u}, \eta \text{ are periodic in time.}$$

Some observations

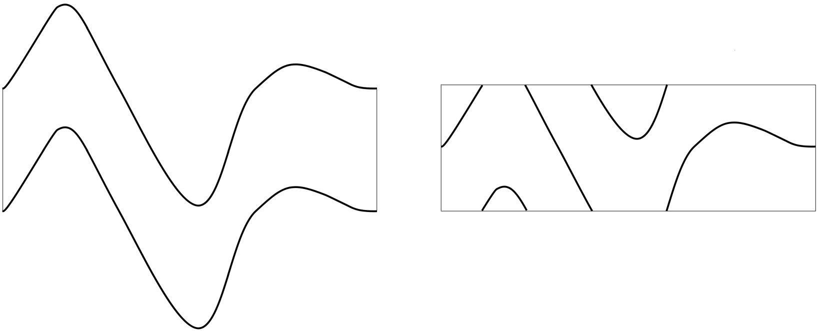
The nature of the studied problem enables us to work with two equivalent formulations of the problem.

- problem in a periodic box as presented above
- problem in a moving domain with η being the lower and upper boundary

For a given $\eta(t, x)$ we introduce an equivalent fluid domain and the corresponding time-space cylinder

$$\begin{aligned}
\Omega^\eta(t) &:= \{(x, z) : x \in (0, L), \eta(t, x) < z < \eta(t, x) + 2H\}, \\
Q_T^\eta &:= \bigcup_{t \in (0,T)} \{t\} \times \Omega^\eta(t).
\end{aligned}$$

Two equivalent domains



Weak solutions

- $\rho \in L^\infty_\#(0,T;L^\gamma_\#(\Omega))$,
  $\mathrm{u} \in L^2_\#(0,T;H^1_\#(\Omega))$,
  $\eta \in L^\infty_\#(0,T;H^2_\#(\Gamma)) \cap H^1_\#(0,T;H^1_{\#,0}(\Gamma))$ with
  $\eta_t \in L^\infty_\#(0,T;L^2_\#(\Gamma))$
- The kinematic coupling $\gamma_{|\hat{\Gamma}^\eta}\mathrm{u} = \eta_t \mathrm{e}_2$ holds on Γ_T.
- The renormalized continuity equation

$$\int_{Q_T} \rho B(\rho)(\partial_t \varphi + \mathrm{u} \cdot \nabla \varphi) = \int_{Q_T} b(\rho)(\nabla \cdot \mathrm{u})\varphi$$

  for all functions $\varphi \in C^\infty_\#(Q_T)$ and any
  $b \in L^\infty(0,\infty) \cap C[0,\infty)$ such that $b(0) = 0$ with
  $B(\rho) = B(1) + \int_1^\rho \frac{b(z)}{z^2} dz$.

Weak solutions II

- The coupled momentum equation

$$\int_{Q_T} \rho \mathrm{u} \cdot \partial_t \boldsymbol{\varphi} + \int_{Q_T} (\rho \mathrm{u} \otimes \mathrm{u}) : \nabla \boldsymbol{\varphi}$$
$$+ \int_{Q_T} \rho^\gamma (\nabla \cdot \boldsymbol{\varphi}) - \int_{Q_T} \mathbb{S}(\nabla \mathrm{u}) : \nabla \boldsymbol{\varphi}$$
$$+ \int_{\Gamma_T} \eta_t \psi_t - \int_{\Gamma_T} \eta_{xx} \psi_{xx} - \int_{\Gamma_T} \eta_{tx} \psi_x$$
$$= - \int_{\Gamma_T} f \psi - \int_{Q_T} \rho \mathrm{F} \cdot \boldsymbol{\varphi}$$

for all $\boldsymbol{\varphi} \in C^\infty_\#(Q_T)$ and all $\psi \in C^\infty_{\#,0}(\Gamma_T)$ such that $\boldsymbol{\varphi}(t, x, \hat{\eta}(t, x)) = \psi(t, x)\mathrm{e}_2$ on Γ_T.

Main theorem

Theorem 1

Let $H, L, T, m_0 > 0$ *be given and let* $\gamma > 1$*. Let* $f \in L^2_\#(\Gamma_T)$ *and* $\mathrm{F} \in L^2_\#(0,T; L^\infty_\#(\Omega))$*. Then there exists at least one solution to the FSI problem such that* $\int_\Omega \rho(t) = m_0$ *for almost all* $t \in (0,T)$*, the energy inequality is satisfied*

$$\int_{Q_T} \mathbb{S}(\nabla \mathrm{u}) : \nabla \mathrm{u} + \int_{\Gamma_T} |\eta_{tx}|^2 \leq \int_{\Gamma_T} f \eta_t + \int_{Q_T} \rho \mathrm{u} \cdot \mathrm{F}$$

and moreover

$$\sup_{t\in(0,T)} \left[\int_\Omega \left(\frac{1}{2}\rho|\mathrm{u}|^2 + \frac{1}{\gamma-1}\rho^\gamma \right) + \int_\Gamma \left(\frac{1}{2}|\eta_t|^2 + \frac{1}{2}|\eta_{xx}|^2 \right) \right] + \int_{Q_T} \mathbb{S}(\nabla \mathrm{u}) : \nabla \mathrm{u} + \int_{\Gamma_T} |\eta_{tx}|^2 \leq C(f, \mathrm{F}, \Omega, m_0).$$

A priori estimates I

Let us assume that we have a sufficiently smooth solution (ρ, u, η) to our problem. The energy associated to the studied system is

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|\mathrm{u}|^2 + \frac{1}{\gamma - 1}\rho^\gamma \right)(t) + \int_\Gamma \left(\frac{1}{2}|\eta_t|^2 + \frac{1}{2}|\eta_{xx}|^2 \right)(t)$$

Further, we denote

$$\mathcal{E} := \sup_{(0,T)} E.$$

The ultimate goal is to show that

$$\mathcal{E} \leq C,$$

where the constant C depends just on the data of the problem.

A priori estimates II

Taking the test functions in the coupled momentum equation as $(\varphi, \psi) = (\mathrm{u}, \eta_t)$ yields

$$\int_{Q_T} \mathbb{S}(\nabla \mathrm{u}) : \nabla \mathrm{u} + \int_{\Gamma_T} |\eta_{tx}|^2 = \int_{\Gamma_T} f \eta_t + \int_{Q_T} \rho \mathrm{u} \cdot \mathrm{F},$$

which then implies

$$||\mathrm{u}||^2_{L^2(0,T;H^1(\Omega))} + ||\eta_t||^2_{L^2(0,T;H^1(\Gamma))} \leq C(1 + \mathcal{E}^\kappa)$$

for some $\kappa > 0$ very small.

A priori estimates IV

Next, we do the same thing but only on time interval (s, t) instead of $(0, T)$. Then we easily get

$$E(t) \leq E(s) + C(1 + \mathcal{E}^{\kappa}) \leq E(s) + C + \kappa\mathcal{E}.$$

Integrating with respect to s and taking supremum over t we end up with

$$\mathcal{E} \leq C_0 \left(1 + \int_0^T E(s)ds\right).$$

In order to close the circle we need

$$\int_0^T E(s)ds \leq \delta_0\mathcal{E} + C(\delta_0)$$

for some $\delta_0 \in (0, \frac{1}{C_0})$.

Here, the main problem lies in the density term. We use the Bogovskii operator on the fixed domain Ω and careful estimates of the arising terms to deduce

$$\int_{Q_T} \rho^\gamma \leq C\left(1+\mathcal{E}^{1-\kappa''}\right)$$

for some $\kappa'' > 0$. With this we can easily close the estimates as follows:

$$\int_0^T E(s)ds \leq C\left(1+\mathcal{E}^{1-\kappa''}\right) \leq C(\delta_0)+\delta_0\mathcal{E}$$

therefore

$$\mathcal{E} \leq C_0\left(1+\int_0^T E(s)ds\right) \leq C_0(1+\delta_0\mathcal{E}+C(\delta_0)).$$

and choosing δ_0 small enough with respect to the constant C_0 we conclude

$$\mathcal{E} \leq C.$$

Approximated problem

The existence of solution is proved via a limit procedure starting from an approximated problem. We introduce

- finite-dimensional spaces both in time and space variables
- we decouple the problem and introduce penalization terms of the type
$$\int_{\Gamma_T} \frac{\eta_t - \mathrm{v} \cdot \mathrm{e}_2}{\varepsilon} \psi$$
in the structure and fluid equations.
- as usual we introduce artificial diffusion in the continuity equation, also with a parameter ε
- we introduce artificial pressure term $\delta \rho^a$ for a large a.
- some other helpful terms multiplied by δ appear in the fluid momentum equation

Limit procedures

- time basis $m \to \infty$
- space basis $n \to \infty$
- penalization and diffusion $\varepsilon \to 0$
- couple back the separated equations
- artificial pressure $\delta \to 0$

Future work

Natural extensions to think about include

- Better boundary conditions, in particular Dirichlet boundary conditions for u on the lateral boundary
- Examine the problem without z-periodicity, maybe under some smallness assumptions?

Thank you

Thank you for your attention.