Linearization of finite elasticity with surface tension

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Introduction – surface energy

Liquids

Surface Cauchy stress - constant

$$\hat{\boldsymbol{\sigma}} = \gamma \hat{\mathbf{I}}$$

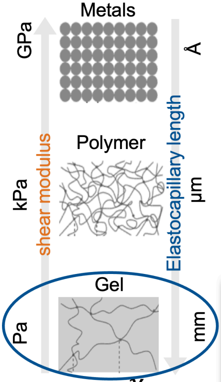
surface tension

Soft Solids

$$\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\Gamma}}(\hat{\mathbf{F}})$$

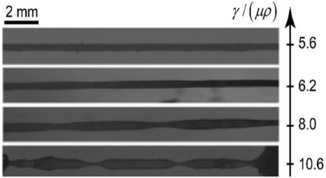
strain dependent
Xu et al., 2011

Solids



$$l_e = \frac{\gamma}{G}$$

Plateau-Rayleigh instability



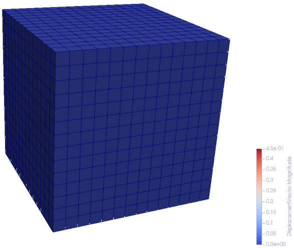
Mora et al. 2010

Modeling soft solids with strain dependent surface energy

Surface/interface elasticity

- Gurtin & Murdoch (1975)
- Gurtin (1986) - interfacial and boundary energy
- Steigmann & Ogden (1997)
- Šilhavý (2011, 2013) - interfacial energy, phase transition
- Javili & Steinmann (2009, 2010) FEM for continua with boundary energies
- Mogilevskaya et al. (2020, 2021) surface elasticity in linearized setting
- Casado Días-Francfort-Lopez Pamies-Mora (2023) elastomers with liquid inclusions

Bulk vs. increasing surface energy (M. Horák, M. Šmejkal)



Functional of elastic energy for nonsimple materials

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open connected Lipschitz set, representing the reference configuration of a hyperelastic body. The elastic-energy functional depending on the deformation field $y : \Omega \to \mathbb{R}^d$ reads

$$\mathcal{G}(y) := \int_\Omega \big(W(\nabla y(x)) + H(\nabla^2 y(x))\big)\, dx - \overline{\mathcal{L}}(y(x) - x) \\ + \gamma \,\|\, |\mathrm{cof}\, \nabla y\, \mathrm{n}| - 1 \,\|_{L^1(\partial\Omega)} \,.$$

Here, W is the stored energy density, H is a convex coercive function representing the energy density depending on the second gradient $\overline{\mathcal{L}}$ is a dead loading linear bounded functional, accounting for the work of given external force fields. Local changes in the surface measure of the boundary of Ω are penalized, as $|\mathrm{cof}\, \nabla y\, \mathrm{n}|$ represents the surface area element of the deformed configuration: $\gamma > 0$, n is the outer unit normal to $\partial\Omega$.

Formal linearization

In the linearization process, we assume that $y(x) = x + \varepsilon v(x)$ for a suitable rescaled displacement v and we set $\mathcal{L} = \overline{\mathcal{L}}/\varepsilon$, which reflects the scaling of external forces. Assuming that H and W are frame indifferent, that H is convex positively p-homogeneous for some suitable $p > 1$ and that W is minimized at the identity with $W(\mathsf{I}) = DW(\mathsf{I}) = 0$, i.e., that identity is the natural state of the body Ω, we consider the rescaled nonlinear global energy

$$\mathcal{G}_\varepsilon(v) := \frac{1}{\varepsilon^2}\int_\Omega W(\mathsf{I} + \varepsilon\nabla v)\,dx + \frac{1}{\varepsilon^p}\int_\Omega H(\varepsilon\nabla^2 v) - \mathcal{L}(v) \\ + \frac{\gamma}{\varepsilon}\,\|\,|\mathrm{cof}\,(\mathsf{I} + \varepsilon\nabla v)\,\mathsf{n}| - 1\,\|_{L^1(\partial\Omega)}.$$

Formal linearization - bulk term

As $\varepsilon \to 0$

$$W(\mathrm{I} + \varepsilon \nabla v) = \frac{\varepsilon^2}{2}\, \mathbb{E}(v)\, \nabla^2 W(\mathrm{I})\, \mathbb{E}(v) + o(\varepsilon^2),$$

the second order term in the Taylor expansion of W produces the standard quadratic potential of linear elasticity (being $D^2 W(\mathrm{I})$ the fourth order elastic tensor), acting by frame indifference only on the symmetric part of the gradient

$$\mathbb{E}(v) := \frac{\nabla v + (\nabla v)^T}{2}.$$

Formal linearization - surface term

On the other hand, by the formula $\mathrm{cof}\,\mathsf{F} = (\det \mathsf{F})\,\mathsf{F}^{-T}$ we have

$$\begin{aligned}\mathrm{cof}\,(\mathsf{I}+\varepsilon\nabla v) &= \det(\mathsf{I}+\varepsilon\nabla v)\,(\mathsf{I}+\varepsilon(\nabla v)^T)^{-1}\\ &= (1+\varepsilon\,\mathrm{div}\,v + o(\varepsilon))\,(\mathsf{I}-\varepsilon(\nabla v)^T + o(\varepsilon))\\ &= \mathsf{I}+\varepsilon\mathbb{A}(v)+o(\varepsilon),\end{aligned}$$

where we have introduced the tensor

$$\mathbb{A}(v) = \mathsf{I}\,\mathrm{div}\,v - (\nabla v)^\top = \mathsf{I}\,\mathrm{tr}\nabla v - (\nabla v)^\top,$$

corresponding to the linearization of the cofactor matrix. As a consequence,

$$|\mathrm{cof}\,(\mathsf{I}+\varepsilon\nabla v)\mathsf{n}| = \sqrt{1+2\varepsilon\,\mathbb{A}(v)\mathsf{n}\cdot\mathsf{n}+o(\varepsilon)} = 1+\varepsilon\,\mathbb{A}(v)\mathsf{n}\cdot\mathsf{n}+o(\varepsilon).$$

Formal limit as $\varepsilon \to 0$

$$\mathcal{G}_*(v) := \frac{1}{2}\int_\Omega \mathbb{E}(v)\, D^2 W(\mathsf{I})\, \mathbb{E}(v)\, dx + \int_\Omega H(\nabla^2 v)\, dx - \mathcal{L}(v) + \gamma \int_{\partial\Omega} |\mathbb{A}(v)\mathsf{n}\cdot\mathsf{n}|\, dS$$

Note that

$$\begin{aligned}\mathbb{A}(v)\mathsf{n}\cdot\mathsf{n} &= (\mathsf{I}\,\mathrm{tr}\nabla v)\mathsf{n}\cdot\mathsf{n} - (\nabla v)^\top\mathsf{n}\cdot\mathsf{n}\\ &= (\mathsf{I}\,\mathrm{tr}\mathbb{E}(v))\mathsf{n}\cdot\mathsf{n} - \mathbb{E}(v)\mathsf{n}\cdot\mathsf{n}\ ,\end{aligned}$$

$$\mathrm{div}\,\mathbb{A}(v) = 0\ ,$$

and

$$\partial^2_{jk} v_i = \partial_j \mathbb{E}(v)_{ik} + \partial_k \mathbb{E}(v)_{ij} - \partial_i \mathbb{E}(v)_{jk},$$

so that $\mathcal{G}_*$ depends only on v, $\mathbb{E}(v)$, and on its gradient.

Another model of surface energy

$$\mathcal{F}(y) := \int_\Omega (W(\nabla y) + H(\nabla^2 y))\, dx - \overline{\mathcal{L}}(y(x) - x)$$
$$+ \gamma \left| \int_{\partial\Omega} |\mathrm{cof}\, \nabla y(\sigma)\, \mathsf{n}(\sigma)|\, dS(\sigma) - |\partial\Omega| \right|.$$

The associated rescaled energies are given by

$$\mathcal{F}_\varepsilon(v) := \frac{1}{\varepsilon^2} \int_\Omega W(\mathsf{I} + \varepsilon\nabla v)\, dx + \frac{1}{\varepsilon^p} \int_\Omega H(\varepsilon\nabla^2 v) - \mathcal{L}(v)$$
$$+ \frac{\gamma}{\varepsilon} \left| \int_{\partial\Omega} |\mathrm{cof}\, (\mathsf{I} + \varepsilon\nabla v)\, \mathsf{n}|\, dS - |\partial\Omega| \right|.$$

We get as the as $\varepsilon \to 0$

$$\left| \int_{\partial\Omega} |\mathrm{cof}\, (\mathsf{I} + \varepsilon\nabla v)\, \mathsf{n}|\, dS - |\partial\Omega| \right| = \varepsilon \left| \int_{\partial\Omega} \mathbb{A}(v)\mathsf{n} \cdot \mathsf{n}\, dS \right| + o(\varepsilon),$$

We notice that the divergence theorem implies, assuming enough smoothness of $\partial\Omega$ and introducing a suitable extension of the normal n to the whole of Ω,

$$\begin{aligned}\int_{\partial\Omega} \mathbb{A}(v)\,\mathsf{n}\cdot\mathsf{n}\,dS &= \int_{\partial\Omega} \mathbb{A}(v)^T\mathsf{n}\cdot\mathsf{n}\,dS \\ &= \int_{\Omega} \operatorname{div}\left(\mathbb{A}(v)^T\mathsf{n}\right)dx = \int_{\Omega} \mathbb{A}(v) : \nabla\mathsf{n}\,dx,\end{aligned}$$

having used the fact that $\mathbb{A}(v)$ is divergence free. Therefore the limit functional $\mathcal{F}_*$ can also be rewritten as

$$\begin{aligned}\mathcal{F}_*(v) = \frac{1}{2}\int_{\Omega} \mathbb{E}(v)\,D^2W(\mathsf{I})\,\mathbb{E}(v)\,dx + \int_{\Omega} H(\nabla^2 v)\,dx - \mathcal{L}(v) \\ + \gamma\left|\int_{\Omega} \mathbb{A}(v) : \nabla\mathsf{n}\,dx\right|.\end{aligned}$$

Assumptions on W and H

$$W : \mathbb{R}^{d\times d} \to [0, +\infty] \text{ is continuous },$$
$$W(\mathsf{RF}) = W(\mathsf{F}) \text{ for every } \mathsf{R} \in SO(d) \text{ and every } \mathsf{F} \in \mathbb{R}^{d\times d},$$
$$W(\mathsf{F}) \geq W(\mathsf{I}) = 0 \text{ for every } \mathsf{F} \in \mathbb{R}^{d\times d},$$
$$W \in C^2(\mathcal{U}) \text{ for some suitable open neighborhood } \mathcal{U} \text{ of } SO(d) \text{ in } \mathbb{R}^{d\times d},$$
$$\mathsf{F}^T D^2 W(\mathsf{I})\,\mathsf{F} \geq C|\mathrm{sym}\,\mathsf{F}|^2 \quad \text{for every } \mathsf{F} \in \mathbb{R}^{d\times d},$$
$$W(\mathsf{F}) \geq \bar{C}\,\mathrm{dist}^2(\mathsf{F}, SO(d)) \quad \text{for every } \mathsf{F} \in \mathbb{R}^{d\times d}.$$

$$H : \mathbb{R}^{d\times d\times d} \to \mathbb{R} \text{ is a convex positively } p\text{-homogeneous function},$$
$$\forall\, \mathsf{B} \in \mathbb{R}^{d\times d\times d} \quad C_0|\mathsf{B}|^p \leq H(\mathsf{B}) \leq C_1(1 + |\mathsf{B}|^p),$$
$$H(\mathsf{RB}) = H(\mathsf{B}) \text{ for every } \mathsf{B} \in \mathbb{R}^{d\times d\times d} \text{ and every } \mathsf{R} \in SO(d).$$

Here the product between R and B is defined as
$(\mathsf{RB})_{imn} = \mathsf{R}_{ik}\mathsf{B}_{kmn}$ for all $i, m, n \in \{1, \ldots, d\}$.

Main results

Theorem (Convergence of minimizers)

Let $p > d/2$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W and H satisfy the assumptions above. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence and let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}_{\Gamma}(\Omega;\mathbb{R}^d)$ be a sequence of minimizers for functionals $\mathcal{G}_{\varepsilon_j}$ over $W^{2,p}_{\Gamma}(\Omega;\mathbb{R}^d)$. Then the sequence $(v_j)_{j\in\mathbb{N}}$ is weakly converging in $W^{2,p}(\Omega;\mathbb{R}^d)$ to the unique solution to the problem

$$\min\left\{\mathcal{G}_*(v) : v \in W^{2,p}_{\Gamma}(\Omega;\mathbb{R}^d)\right\}.$$

Theorem

Let $p > d/2$. Let W and H satisfy the assumptions above. Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. If $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega,\mathbb{R}^d)$ is a sequence of minimizers of $\mathcal{G}_{\varepsilon_j}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$, then there exists a sequence $(\mathrm{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ such that, by defining

$$u_j(x) := \mathrm{R}_j^T v_j(x) + \frac{1}{\varepsilon_j}(\mathrm{R}_j^T x - x),$$

in the limit as $j \to +\infty$, along a suitable (not relabeled) subsequence, there holds

$$\nabla u_j \to \nabla u_* \ \ \text{weakly in } W^{1,p}(\Omega;\mathbb{R}^{d\times d}),$$

where $u_* \in W^{2,p}(\Omega,\mathbb{R}^d)$ is a minimizer of $\overline{\mathcal{G}}$ over $W^{2,p}(\Omega,\mathbb{R}^d)$, and

$$\mathcal{G}_{\varepsilon_j}(v_j) \to \overline{\mathcal{G}}(u_*), \qquad \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \mathcal{G}_{\varepsilon_j} \to \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \overline{\mathcal{G}}.$$

Main results

Lemma (Γ-**limsup**)

Let $p > d/2$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W and H satisfy the assumptions above. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $v \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$. There exists a sequence $(v_j)_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d) \cap W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ such that

$$v_j \to v \quad \text{strongly in } W^{2,p}(\Omega;\mathbb{R}^d) \text{ as } j \to +\infty$$

and

$$\lim_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j) = \mathcal{G}_*(v).$$

Main results

Lemma (Γ-liminf)

Let $p > d/2$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W and H satisfy the assumptions above. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $v \in W^{2,p}_{\Gamma}(\Omega;\mathbb{R}^d)$. Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}_{\Gamma}(\Omega;\mathbb{R}^d)$ be a sequence that weakly converges to v in $W^{2,p}(\Omega;\mathbb{R}^d)$. Then

$$\liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j) \geq \mathcal{G}_*(v).$$

Key lemma

The following is a key lemma, providing the rigorous linearization of the interfacial term.

Lemma

Let $p > d/2$. Let $v \in W^{2,p}(\Omega; \mathbb{R}^d)$. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence and let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega; \mathbb{R}^d)$ be a sequence such that ∇v_j weakly converge to ∇v in $W^{1,p}(\Omega; \mathbb{R}^{d\times d})$ as $j \to +\infty$. Then

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j} \int_{\partial\Omega} \left| |\mathrm{cof}\,(\mathrm{I} + \varepsilon_j \nabla v_j)\,\mathrm{n}| - 1 \right| \, dS = \int_{\partial\Omega} |\mathbb{A}(v)\mathrm{n} \cdot \mathrm{n}| \, dS \ .$$

Important steps of the proof

By the Cayley-Hamilton formula, we have

$$\mathrm{cof}\,(\mathsf{I} + \varepsilon_j \nabla v_j) = \mathsf{I} + \varepsilon_j \mathbb{A}(v_j) + \sum_{k=2}^{d-1} \varepsilon_j^k\, \mathbb{B}_k(v_j)$$

where $\mathbb{B}_k(v_j)$ is a matrix whose entries are polynomials of degree k in the entries of ∇v_j, and the sum is understood to be zero if $d = 2$. Letting

$$\mathbb{B}(v_j) := \sum_{k=2}^{d-1} \varepsilon_j^{k-2} \mathbb{B}_k(v_j),$$

we get therefore

$$\mathrm{cof}\,(\mathsf{I} + \varepsilon_j \nabla v_j) = \mathsf{I} + \varepsilon_j \mathbb{A}(v_j) + \varepsilon_j^2\, \mathbb{B}(v_j).$$

We notice that

$$\begin{aligned} |\mathrm{cof}\,(\mathsf{I} + \varepsilon_j \nabla v_j)\,\mathsf{n}|^2 &= |\mathsf{n} + \varepsilon_j \mathbb{A}(v_j)\,\mathsf{n} + \varepsilon_j^2\, \mathbb{B}(v_j)\,\mathsf{n}|^2 \\ &= 1 + 2\varepsilon_j \mathbb{A}(v_j)\,\mathsf{n}\cdot\mathsf{n} + \varepsilon_j^2\, \mathbb{D}(v_j)\,\mathsf{n}\cdot\mathsf{n}, \end{aligned}$$

where

$$\mathbb{D}(v_j) := \mathbb{A}(v_j)^T \mathbb{A}(v_j) + 2\mathbb{B}(v_j) + 2\varepsilon_j \mathbb{B}(v_j)^T \mathbb{A}(v_j) + \varepsilon_j^2 \mathbb{B}(v_j)^T \mathbb{B}(v_j).$$

We observe that the following properties hold:

$$\nabla v_j \to \nabla v \text{ strongly in } L^1(\partial\Omega; \mathbb{R}^{d\times d}) \text{ as } j \to +\infty,$$

$$\mathbb{A}(\nabla v_j) \to \mathbb{A}(\nabla v) \text{ strongly in } L^1(\partial\Omega; \mathbb{R}^{d\times d}) \text{ as } j \to +\infty,$$

$$\text{the sequence } (\mathbb{B}(v_j))_{j\in\mathbb{N}} \text{ is bounded in } L^1(\partial\Omega; \mathbb{R}^{d\times d}).$$

We define

$$Q_j := \{x \in \partial\Omega : |\mathbb{A}(v_j(x))| + \varepsilon_j |\mathbb{B}(v_j(x))| < 2^{-4} \varepsilon_j^{-1/4}\}$$

and we notice that

$$\begin{aligned} S(\partial\Omega \setminus Q_j) &\leq \int_{\partial\Omega \setminus Q_j} 16 \varepsilon_j^{1/4} (|\mathbb{A}(v_j)| + \varepsilon_j |\mathbb{B}(v_j)|) \, dS \\ &\leq 16 \varepsilon_j^{1/4} \int_{\partial\Omega} (|\mathbb{A}(v_j)| + \varepsilon_j |\mathbb{B}(v_j)|) \, dS \end{aligned}$$

so that $S(\partial\Omega \setminus Q_j) \to 0$ as $j \to +\infty$.

On Q_j it holds

$$|2\varepsilon_j\,\mathbb{A}(v_j)\,\mathrm{n}\cdot\mathrm{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathrm{n}\cdot\mathrm{n}| \le 2\varepsilon_j\,|\mathbb{A}(v_j)|+\varepsilon_j^2\,|\mathbb{D}(v_j)| < \frac{1}{2}\varepsilon_j^{3/4} < \frac{1}{2},$$

$$\varepsilon_j|\mathbb{D}(v_j)| \le \sqrt{\varepsilon_j} + 2\varepsilon_j|\mathbb{B}(v_j)|.$$

$$|\mathrm{cof}\,(\mathrm{I}+\varepsilon_j\nabla v_j)\,\mathrm{n}|-1 = \sqrt{1+2\varepsilon_j\mathbb{A}(v_j)\,\mathrm{n}\cdot\mathrm{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathrm{n}\cdot\mathrm{n}}-1$$

$$= \varepsilon_j\mathbb{A}(v_j)\,\mathrm{n}\cdot\mathrm{n}+\frac{\varepsilon_j^2}{2}\mathbb{D}(v_j)\,\mathrm{n}\cdot\mathrm{n}+\sum_{k=2}^{+\infty}\alpha_k(2\varepsilon_j\mathbb{A}(v_j)\,\mathrm{n}\cdot\mathrm{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathrm{n}\cdot\mathrm{n})^k,$$

where $\alpha_k := \dfrac{(-1)^{k-1}(2k)!}{4^k(k!)^2(2k-1)}$ and

$$\sum_{k=2}^{+\infty} \alpha_k \left(2\varepsilon_j \mathbb{A}(v_j)\,\mathsf{n}\cdot\mathsf{n} + \varepsilon_j^2\, \mathbb{D}(v_j)\,\mathsf{n}\cdot\mathsf{n}\right)^k \to 0$$

We have

$$1_{Q_j}\, \frac{|\mathrm{cof}\,(\mathsf{I}+\varepsilon_j \nabla v_j)\,\mathsf{n}| - 1}{\varepsilon_j} = 1_{Q_j}\, \mathbb{A}(v_j)\,\mathsf{n}\cdot\mathsf{n} + 1_{Q_j}\, \frac{\varepsilon_j}{2} \mathbb{D}(v_j)\,\mathsf{n}\cdot\mathsf{n}$$

$$+ 1_{Q_j}\, \frac{1}{\varepsilon_j} \sum_{k=2}^{+\infty} \alpha_k (2\varepsilon_j \mathbb{A}(v_j)\,\mathsf{n}\cdot\mathsf{n} + \varepsilon_j^2\, \mathbb{D}(v_j)\,\mathsf{n}\cdot\mathsf{n})^k,$$

Since $S(\partial\Omega \setminus Q_j) \to 0$ we deduce that

$$1_{Q_j}\, \frac{|\mathrm{cof}\,(\mathsf{I}+\varepsilon_j \nabla v_j)\,\mathsf{n}| - 1}{\varepsilon_j} \to \mathbb{A}(v)\,\mathsf{n}\cdot\mathsf{n} \text{ strongly in } L^1(\partial\Omega) \text{ as } j \to +\infty$$

so that

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j} \int_{Q_j} \left||\mathrm{cof}\,(\mathsf{I}+\varepsilon_j \nabla v_j)\,\mathsf{n}| - 1\right| dS = \int_{\partial\Omega} |\mathbb{A}(v)\,\mathsf{n}\cdot\mathsf{n}|\, dS.$$

The proof concludes by showing that

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j} \int_{\partial\Omega\setminus Q_j} \left||\mathrm{cof}\,(\mathsf{I}+\varepsilon_j \nabla v_j)\,\mathsf{n}| - 1\right| dS = 0.$$

Take home messages

1. In many cases, surface elasticity plays an important role.
2. We introduced two model examples of surface elastic energy
3. Results for pure traction problems are also available if

$$\mathcal{L}(a + \mathrm{M}x) = 0 \text{ for every } a \in \mathbb{R}^d \text{ and every } \mathrm{M} = -\mathrm{M}^\top.$$

4. Preprint available at arXiv:2312.08783

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Thank you for your attention!