On unsteady flows of viscoelastic fluids of Giesekus type

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Praha, 23th September 2024

Viscoelastic materials: Models P_λ

- Motivation: Analysis of flows of viscoelastic materials
- Biomaterials, geomaterials, synthetic rubbers
- Complex structure: More relaxation mechanisms

Find $(\boldsymbol{v}, p, \mathbb{F}_1, \mathbb{F}_2)$ satisfying in $Q_T := (0,T) \times \Omega$

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0, \\
\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{D} + \nabla p &= \operatorname{div}(G_1 \mathbb{B}_1 + G_2 \mathbb{B}_2), \\
\overset{\triangledown}{\mathbb{B}}_i + \left(\mathbb{B}_i^{2-\lambda_i} - \mathbb{B}_i^{1-\lambda_i} \right) &= \mathbb{O} \qquad i = 1, 2, \\
\mathbb{B}_i &= \mathbb{F}_i \mathbb{F}_i^T \quad i = 1, 2, \\
\det \mathbb{F}_i &> 0 \qquad i = 1, 2,
\end{aligned}
$$

with boundary condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T := (0,T) \times \partial\Omega$$

and initial conditions

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \;\; \mathbb{F}_i(0,\cdot) = \mathbb{F}_{i_0}, \;\; \mathbb{B}_i(0,\cdot) = \mathbb{F}_{i_0} \mathbb{F}_{i_0}^T \quad \text{in } \Omega.$$

Viscoelastic materials: Models P_λ

- Second order models
- $\lambda_1 = \lambda_2 = 1 \to$ two Oldroyd-B models ($\approx$ Burgers model)
- $\lambda_1 = \lambda_2 = 0 \to$ two Giesekus models

All models: same forms of Helmholtz free energy

$$\psi := \sum_{i=1}^{2} G_i(\mathrm{tr}\, \mathbb{B}_i - d - \ln\det \mathbb{B}_i).$$

Rate of entropy production: differs by values of λ_1, λ_2

$$\xi := |\mathbb{D}|^2 + \sum_{i=1}^{2} \frac{G_i}{2} |\mathbb{B}_i^{\frac{2-\lambda_i}{2}}(\mathbb{I} - \mathbb{B}_i^{-1})|^2.$$

Energy identity:

$$\int_\Omega \left(|\boldsymbol{v}(t)|^2 + \psi(t)\right) + 2\int_0^t \int_\Omega \xi = \int_\Omega \left(|\boldsymbol{v}(0)|^2 + \psi(0)\right).$$

Burgers-Giesekus – 3D weak formulation

Let

- $\Omega \subset \mathbb{R}^3$ bounded domain, $\Omega \in C^{0,1}$, $i \in \{1, 2\}$
- $\boldsymbol{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$, $\mathbb{F}_{i_0} \in (L^2(\Omega))^{3\times 3}$, $\mathbb{B}_{i_0} := \mathbb{F}_{i_0}\mathbb{F}_{i_0}^T$, $\det \mathbb{F}_{i_0} > 0$, $\ln \det \mathbb{F}_{i_0} \in L^1(\Omega)$.

Weak solution to Burgers-Giesekus model is $(\mathbf{v}, \mathbb{F}_1, \mathbb{F}_2)$

- $\boldsymbol{v} \in C_{weak}\left([0,T]; L^2_{\mathbf{n},\mathrm{div}}\right) \cap L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^{\frac{4}{3}}(0,T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$
- $\mathbb{F}_i \in C\left([0,T]; (L^2(\Omega))^{3\times 3}\right) \cap \left(L^{4+2\mu_i}(Q_T)\right)^{3\times 3}$, $\partial_t \mathbb{F}_i \in L^{\frac{4}{3}}(0,T; (W^{1,2})^*)$
- $\mathbb{B}_i \in C\left([0,T]; (L^1(\Omega))^{3\times 3}\right) \cap \left(L^{2+\mu_i}(Q_T)\right)^{3\times 3}$, $\partial_t \mathbb{B}_i \in L^1(0,T; (W^{1,4})^*)$
- $\mathbb{B}_i = \mathbb{F}_i\mathbb{F}_i^T$, $\det \mathbb{F}_i > 0$ a.e. in Q_T

$$\langle \partial_t \boldsymbol{v}, \mathbf{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \int_\Omega \sum_{i=1}^{2} G_i \mathbb{B}_i : \nabla \boldsymbol{w} = 0$$

$$\langle \partial_t \mathbb{B}_i, \mathbb{A}\rangle - \int_\Omega (\mathbb{B}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega (\nabla \boldsymbol{v}\mathbb{B}_i + \mathbb{B}_i(\nabla \boldsymbol{v})^T) : \mathbb{A} + \int_\Omega (\mathbb{B}_i^{2+\mu_i} - \mathbb{B}_i^{1+\mu_i}) : \mathbb{A} = 0$$

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Mathematical results – Global weak solutions

- Oldroyd-B model – open
- Lions, Masmoudi (2000): Oldroyd-B, instead of $\overset{\triangledown}{\mathbb{B}}$

$$\partial_t \mathbb{B} + \sum_{j=1}^{d} \partial_{x_j}(v_k \mathbb{B}) - \mathbb{W}\mathbb{B} - \mathbb{B}\mathbb{W}^T, \quad \mathbb{W} := \frac{1}{2}\left(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T\right)$$

- Masmoudi (2011): Weak sequential stability of weak solutions to Giesekus model ($\mu = 0$)
  - brief outline of the proof (reference to analysis of other models)
  - we found inconsistencies
  - not introduced approximations

- Bulíček, Los, Lu, Málek (2022): Existence of weak solutions to mixture of two Giesekus models in 2D
  - new existence result for a second order model
  - rigorous proof with full details

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Key ideas/steps

First idea (Masmoudi):

- Consider equations for $\mathbb{F}$

$$\partial_t \mathbb{F} + \operatorname{div}(\mathbb{F} \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{F} + \frac{1}{2}\left((\mathbb{F}\mathbb{F}^T)^{1+\mu} - (\mathbb{F}\mathbb{F}^T)^{\mu}\right)\mathbb{F} = \mathbb{O}. \quad (1)$$

- Sum of (1) multiplied by $\mathbb{F}^T$ from right with transposed (1) multiplied by $\mathbb{F}$ from left

$$\partial_t \mathbb{B} + \mathrm{Div}(\mathbb{B} \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{B} - \mathbb{B}(\nabla \boldsymbol{v})^T + (\mathbb{B}^{2+\mu} - \mathbb{B}^{1+\mu}) = \mathbb{O}, \quad \mathbb{B} = \mathbb{F}\mathbb{F}^T.$$

- Advantages: $\mathbb{F}$ better integrability than $\mathbb{B} \Rightarrow$ larger set of test functions
  - increases chance to get compactness of approximations
  - short proof of $\det \mathbb{F} > 0$
- we get $\mathbb{B} = \mathbb{F}\mathbb{F}^T$ directly

Key ideas/steps

Approximations:

- Let $(\boldsymbol{v}_k, \mathbb{F}_k)_{k\in\mathbb{N}}$ be sequence of weak solutions to balance of momentum and (1).
- Uniform bounds for $(\boldsymbol{v}_k, \partial_t \boldsymbol{v}_k, \nabla \boldsymbol{v}_k, \mathbb{F}_k, \partial_t \mathbb{F}_k)$ + Aubin-Lions lemma $\Rightarrow$ extract subsequences $\boldsymbol{v}_k \to \boldsymbol{v}$, $\mathbb{F}_k \to \mathbb{F}$ as $k \to \infty$ such that

$$\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbb{D} + \nabla p - \operatorname{div} \overline{\mathbb{F}\mathbb{F}^T} &= \mathbf{0} \\
\partial_t \mathbb{F} + \operatorname{div}(\mathbb{F} \otimes \boldsymbol{v}) - \overline{(\nabla \boldsymbol{v})\mathbb{F}} + \frac{1}{2}\overline{((\mathbb{F}\mathbb{F}^T)^{1+\mu} - (\mathbb{F}\mathbb{F}^T)^{\mu})\mathbb{F}} &= \mathbb{O}
\end{aligned}$$

We need

- $\overline{(\nabla \boldsymbol{v})\mathbb{F}} = (\nabla \boldsymbol{v})\mathbb{F}$
- $\overline{\mathbb{F}\mathbb{F}^T} = \mathbb{F}\mathbb{F}^T$
- $\overline{(\mathbb{F}\mathbb{F}^T)^{\tilde{\mu}}\mathbb{F}} = (\mathbb{F}\mathbb{F}^T)^{\tilde{\mu}}\mathbb{F}$

It suffices

- $\mathbb{F}_k \to \mathbb{F}$ in $(L^2(Q_T))^{3\times 3}$

Main ideas of the proof

Proof of compactness: 3 main steps

1. For all $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, $\varphi \geq 0$

$$-\int_\Omega |\mathbb{F}_0|^2 \varphi(0) - \int_{Q_T} (\overline{|\mathbb{F}|^2} \partial_t \varphi + \overline{|\mathbb{F}|^2} \boldsymbol{v} \cdot \nabla \varphi) - \left\langle 2\overline{\nabla \boldsymbol{v} : \mathbb{F}\mathbb{F}^T} - \overline{(\mathbb{F}\mathbb{F}^T)^{1+\mu} : \mathbb{F}\mathbb{F}^T} + \overline{(\mathbb{F}\mathbb{F}^T)^{\mu} : \mathbb{F}\mathbb{F}^T}, \varphi \right\rangle = 0$$

$$-\int_\Omega |\mathbb{F}_0|^2 \varphi(0) - \int_{Q_T} (|\mathbb{F}|^2 \partial_t \varphi + |\mathbb{F}|^2 \boldsymbol{v} \cdot \nabla \varphi) - \int_{Q_T} \left(2\overline{\nabla \boldsymbol{v}\mathbb{F}} : \mathbb{F} - \overline{(\mathbb{F}\mathbb{F}^T)^{1+\mu}\mathbb{F}} : \mathbb{F} + \overline{(\mathbb{F}\mathbb{F}^T)^{\mu}\mathbb{F}} : \mathbb{F} \right) \varphi = 0$$

2. From Step 1 derive for all $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, $\varphi \geq 0$

$$-\int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right) \partial_t \varphi - \int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right) \boldsymbol{v} \cdot \nabla \varphi \leq \int_{Q_T} L \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right) \varphi,$$

where $L \in L^2(Q_T)$ is a suitable function.

3. Conclude $\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2$ a.e. in Q_T, which is equivalent to

$$\mathbb{F}_k \to \mathbb{F} \text{ in } (L^2(Q_T))^{3\times 3}.$$

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