Qualitative/quantitative homogenization of some non-Newtonian flows

Yong Lu

Department of Mathematics, Nanjing University

Modelling, PDE analysis and computational mathematics in materials science
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Homogenization problems in fluid mechanics

Homogenization problems in fluid mechanics represent the study of the asymptotic behavior of fluid flows in perforated domains as the number of holes goes to infinity and the size of holes goes to zero simultaneously.

Equations:

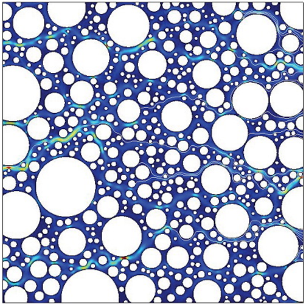
- Stokes equations, Navier-Stokes equations, Euler equations.
- Incompressible, Compressible.
- Stationary, Non-stationary.

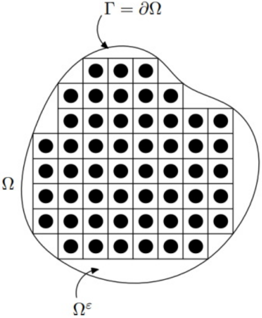
Parameters:

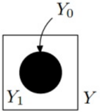
- Mutual distance between holes ε.
- Size of holes a_ε. Two typical cases: $a_\varepsilon \sim \varepsilon^\alpha$, or $a_\varepsilon \sim e^{-\sigma\varepsilon^{-\alpha}}$.
- Dimensions: $d = 2,\ 3$.

Perforated media

Perforated domains.







Periodically perforated domain:

$$\Omega_\varepsilon := \Omega \setminus (\cup_{k\in\mathbb{Z}^d} T_{\varepsilon,k}), \;\; T_{\varepsilon,k} := \varepsilon k + a_\varepsilon T.$$

A beginning result by L. Tartar

L. Tartar, 1980. Stokes equations, $d = 3$, $a_\varepsilon \sim \varepsilon$, $\mathrm{f} \in L^2(\Omega; \mathbb{R}^3)$:

$$\begin{cases} -\Delta \mathrm{u}_\varepsilon + \nabla p_\varepsilon = \mathrm{f}, & \text{in } \Omega_\varepsilon, \\ \operatorname{div} \mathrm{u}_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \mathrm{u}_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1}$$

Result: $\mathrm{u}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$, $\quad p_\varepsilon \in L_0^2(\Omega_\varepsilon)$,

$$\left(\frac{\tilde{\mathrm{u}}_\varepsilon}{\varepsilon^2}, \tilde{p}_\varepsilon \right) \to (\mathrm{u}, p) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^3) \times L_0^2(\Omega),$$

where the limit couple (u, p) satisfies the Darcy's law:

$$\mathrm{u} = M_0^{-1}(\mathrm{f} - \nabla p), \ \operatorname{div} \mathrm{u} = 0 \text{ in } \Omega; \quad \mathrm{u} \cdot \mathrm{n} = 0 \text{ on } \partial\Omega.$$

M_0 is a positive definite matrix determined by the distribution of holes.

A beginning result by L. Tartar

Why ε^2? Poincaré inequality and L^2 estimate gives:

$$\|\nabla \mathrm{u}\|^2_{L^2(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega_\varepsilon)}\|\mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\|f\|_{L^2(\Omega_\varepsilon)}\|\nabla \mathrm{u}\|_{L^2(\Omega_\varepsilon)}.$$

Poincaré inequality in porous media: given $\mathrm{u} \in W^{1,2}_0(\Omega_\varepsilon)$:

$$\|\mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla \mathrm{u}\|_{L^2(\Omega_\varepsilon)}$$

Thus

$$\|\nabla \mathrm{u}\|^2_{L^2(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega_\varepsilon)}\|\mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|f\|_{L^2(\Omega_\varepsilon)}\|\nabla \mathrm{u}\|_{L^2(\Omega_\varepsilon)}$$

implying

$$\|\nabla \mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon, \quad \|\mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla \mathrm{u}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^2.$$

An observation by Cioranescu and Murat

D. Cioranescu, F. Murat, 1982. *Un terme étrange venu d'ailleurs*. Poisson equations, $d = 2$, $a_\varepsilon \sim e^{-\sigma\varepsilon^{-2}}$, $f \in L^2(\Omega)$:

$$\begin{cases} -\Delta u_\varepsilon = f, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2}$$

Result: $u_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$,

$$\tilde{u}_\varepsilon \to u \quad \text{weakly in } W_0^{1,2}(\Omega),$$

where the limit u satisfies:

$$\begin{cases} -\Delta u + \sigma^{-1} u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Systematic study by Allaire

G. Allaire, 1991, Stokes equations, $d \geq 2$, $f \in L^2(\Omega; \mathbb{R}^d)$:

$$\begin{cases} -\Delta \mathrm{u}_\varepsilon + \nabla p_\varepsilon = \mathrm{f}, & \text{in } \Omega_\varepsilon, \\ \operatorname{div} \mathrm{u}_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \mathrm{u}_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{4}$$

The homogenization process is determined by the ratio:

$$\sigma_\varepsilon := \varepsilon^{-d} a_\varepsilon^{d-2}, \quad d \geq 3; \qquad \sigma_\varepsilon := \varepsilon^{-2} \left|\log a_\varepsilon\right|, \quad d = 2.$$

- If $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, Stokes $\to$ Stokes.
- If $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \infty$, Stokes $\to$ Darcy's law.
- If $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0$, Stokes $\to$ Brinkman's law.

Other results on Poisson and Stokes equations

- Hillairet 2018. Hillairet-Mecherbet 2018, Hillairet-Carrapatoso 2020, Method of reflection. Stokes problem with nonzero boundary values on the holes, randomly distributed spheres. No periodicity.
- Höfer-Velázquez 2018. Method of reflection. Poisson and Stokes. No periodicity. $\mathbb{R}^d$.
- Giunti-Höfer, 2019, 2020. Poisson and Stokes. Randomly distributed spherical holes.
- Hillairet-Wu 2019, Gérard-Varet & Hillairet 2020, Gérard-Varet & Höfer 2021, Gérard-Varet & Mecherbet 2022. Einstein's formula on effective viscosity of dilute suspensions. 1st order and 2nd order expansion.
- Jing 2020: Poisson, elasticity, a unified approach, convergence rates.
- L. 2020, 2021: Stokes, a unified approach, bounded domains and $\mathbb{R}^d$, $d \geq 2$.
- Shen 2020, 2021, 2022: Stokes, $a_\varepsilon \sim \varepsilon$, sharp convergence rate, large scale regularity, uniform $W^{m,q}$ estimates.
- Wang-Xu-Zhang, 2022, unsteady Stokes, Darcy's law with memory, convergence rates.
- Jing-L.-Prange 2024: Stokes, quantitative estimates.

For Navier-Stokes equations

Incompressible Navier-Stokes equations:

$$\partial_t \mathrm{u} + \mathrm{u} \cdot \nabla \mathrm{u} - \nu \Delta \mathrm{u} + \nabla p = \mathrm{f}, \quad \mathrm{div}\, \mathrm{u} = 0.$$

Compressible Navier-Stokes equations:

$$\partial_t \varrho + \mathrm{div}\,(\varrho \mathrm{u}) = 0, \quad \partial_t(\varrho \mathrm{u}) + \mathrm{div}\,(\varrho \mathrm{u} \otimes \mathrm{u}) + \nabla p(\varrho) = \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathrm{u}) + \varrho \mathrm{f},$$

$$\mathbb{S}(\nabla_x \mathrm{u}) = \mu^S \left(\nabla_x \mathrm{u} + \nabla_x^{\mathrm{T}} \mathrm{u} - \frac{2}{d} (\mathrm{div}_x \mathrm{u}) \mathbb{I} \right) + \mu^B (\mathrm{div}_x \mathrm{u}) \mathbb{I}, \quad p(\varrho) \sim \varrho^\gamma.$$

Some results on INS:

- G. Allaire, 1991. Stationary.
- A. Mikelić, 1991, evolutionary, $d \geq 2$, $a_\varepsilon \sim \varepsilon$. Darcy's law.
- Feireisl-Namlyeyeva-Nečasová 2016, evolutionary, critical size.
- L.-Yang 2022: evolutionary: small holes and large holes.

For Navier-Stokes equations

Some results on CNS

- Masmoudi, 2002. Non-stationary compressible Navier-Stokes. $d = 2, 3$, $a_\varepsilon \sim \varepsilon$, $p(\varrho) \sim \varrho^\gamma$, $\gamma \geq d$. Darcy's law and PME.
- Feireisl–Novontný–Takahashi, 2010. For the complete Navier-Stokes-Fourier system. $a_\varepsilon \sim \varepsilon$, PME.
- Höfer-Kowalczyk-Schwarzacher 2021, homogenization and low-Mach number limit. Darcy's law
- Bella-Oschmann, 2021, 2022, randomly perforated domains with small holes; low-Mach number limit with critical size of holes.
- Nečasova-Oschmann 2023, Oschmann-Pokorny 2023, evolutionary Navier-Stokes(-Fourier) with very tiny holes.
- Höfer-Nečasova-Oschmann 2024: $a_\varepsilon \sim \varepsilon$, convergence rates from compressible Navier-Stokes to Darcy'a law.

For Navier-Stokes equations with $a_\varepsilon \ll \varepsilon$.

Previous studies : $a_\varepsilon \sim \varepsilon$.

How about: $a_\varepsilon \ll \varepsilon$? For example: $a_\varepsilon = \varepsilon^\alpha,\ \alpha > 1$.

Theorem (Feireisl-L. 2015)

Let $d = 3$ and $a_\varepsilon \sim \varepsilon^\alpha$. If $\alpha > 3$ and $\gamma \geq 3$, then:

$$\text{stationary compressible Navier-Stokes equations in } \Omega_\varepsilon$$
$$\to \text{stationary compressible Navier-Stokes equations in } \Omega.$$

Theorem (Diening-Feireisl-L. 2017)

Let $d = 3$ and $a_\varepsilon \sim \varepsilon^\alpha$. Suppose $2 < \gamma \leq 3$ and $\alpha > 3$ be given such that $\frac{\gamma-2}{2\gamma-3} \cdot \alpha > 1$. Then:

$$\text{stationary compressible Navier-Stokes equations in } \Omega_\varepsilon$$
$$\to \text{stationary compressible Navier-Stokes equations in } \Omega.$$

For Navier-Stokes equations with $a_\varepsilon \ll \varepsilon$.

Theorem (Schwarzacher-L., 2018)

Let $d = 3$ and $\gamma > 6,\ \alpha > 3$, such that $\frac{\gamma-6}{2\gamma-3} \cdot \alpha > 3$. Then:

evolutionary compressible Navier-Stokes equations in Ω_ε
$\to$ evolutionary compressible Navier-Stokes equations in Ω.

Theorem (Pokorny-L., 2021)

Let $d = 3$, $p(\varrho) \sim \varrho^\gamma$, $\kappa(\theta) \sim 1 + \theta^m$, with

$$\alpha > 3,\ m > 2,\ \gamma > 2,\ \alpha > \max\left\{\frac{2\gamma-3}{\gamma-2}, \frac{3m-2}{m-2}\right\}.$$

stationary compressible Navier-Stokes-Fourier equations in Ω_ε
$\to$ stationary compressible Navier-Stokes-Fourier equations in Ω.

Homogenization of non-Newtonian flows: setting

Perforated domain:

$$\Omega_\varepsilon := \Omega \setminus (\cup_{k\in\mathbb{Z}^d} T_{\varepsilon,k}).$$

Here $T_{\varepsilon,k} := \varepsilon k + a_\varepsilon T$ with $a_\varepsilon = \varepsilon^\alpha, \alpha \geq 1$.

Non-Newtonian flows of Carreau-Yasuda type:

$$\begin{cases} \partial_t \mathrm{u}_\varepsilon + (\mathrm{u}_\varepsilon \cdot \nabla)\mathrm{u}_\varepsilon - \mathrm{div}\{\eta_r(D\mathrm{u}_\varepsilon)D\mathrm{u}_\varepsilon\} + \nabla p_\varepsilon = f, & \text{in } (0,T)\times\Omega_\varepsilon \\ \mathrm{div}\,\mathrm{u}_\varepsilon = 0, & \text{in } (0,T)\times\Omega_\varepsilon \\ \mathrm{u}_\varepsilon = 0, & \text{on } (0,T)\times\partial\Omega_\varepsilon \\ \mathrm{u}_\varepsilon|_{t=0} = \mathrm{u}_0, & \text{in } \Omega_\varepsilon \end{cases} \tag{5}$$

- $D\mathrm{u}_\varepsilon = \frac{1}{2}(\nabla \mathrm{u}_\varepsilon + \nabla^T \mathrm{u}_\varepsilon)$.
- $f \in L^\infty(\Omega;\mathbb{R}^3)$, $\mathrm{u}_0 \in L^2(\Omega;\mathbb{R}^3)$.

Homogenization of non-Newtonian flows: setting

Carreau-Yasuda law:

$$\eta_r(D\mathrm{u}_\varepsilon) = (\eta_0 - \eta_\infty)(1 + \kappa|D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1} + \eta_\infty, \quad \eta_0 \geq \eta_\infty > 0,\ \lambda > 0,\ r > 1.$$

- $r = 2$: Newtonian.
- $r < 2$: pseudo-plastic, shear-thinning.
- $r > 2$: dilatant, shear-thickening, concentrated suspensions.

Existence of finite energy weak solutions in $3D$ with $\mathrm{u}_\varepsilon \in L^2(0, T; W^{1,2}_{0,\mathrm{div}}(\Omega_\varepsilon)) \cap L^r(0, T; W^{1,r}_{0,\mathrm{div}}(\Omega_\varepsilon)) \cap C_{\mathrm{w}}([0, T]; L^2(\Omega_\varepsilon))$:

- Ladyzhenskaya 1969 with $r \geq 11/5$, uniqueness with $r \geq 5/2$.
- Diening, Ružička, and Wolf 2010, Bulíček, Gwiazda, Málek, and Swierczewska-Gwiazda 2012 : for $r > 6/5$.
- With Carreau-Yasuda law, due to the presence of Newtonian part of the stress tensor (i.e., $\eta_\infty > 0$): $r > 1$.

Homogenization of non-Newtonian flows: known results

The previous results are mainly focuses on the case $\alpha = 1$:

- Bourgeau-Mickelić 1996: Stationary case, Darcy's law.
- L.-Qian 2023: Evolutionary case, $1 < r < \infty$, Darcy's law :

$$\varepsilon^{-2}\tilde{\mathrm{u}}_\varepsilon \to \mathrm{u} \text{ weakly in } L^2((0,T)\times\Omega),$$

with

$$\frac{1}{2}\eta_0 \mathrm{u} = M_0^{-1}(\mathrm{f} - \nabla p), \quad \operatorname{div} \mathrm{u} = 0 \text{ in } (0,T)\times\Omega,$$
$$\mathrm{u}\cdot\mathrm{n} = 0 \text{ on } (0,T)\times\partial\Omega.$$

Poincaré inequality in porous media: $1 \le q < 3, \alpha \ge 1$

$$\|\mathrm{u}\|_{L^q(\Omega_\varepsilon)} \le C \min\left\{\varepsilon^{\frac{3-(3-q)\alpha}{q}}, 1\right\}\|\nabla \mathrm{u}\|_{L^q(\Omega_\varepsilon)}, \ \forall\, \mathrm{u} \in W_0^{1,q}(\Omega_\varepsilon).$$

Homogenization of non-Newtonian flows

Perforated domain with $\alpha > 1$:

$$\Omega_\varepsilon := \Omega \setminus (\cup_{k\in\mathbb{Z}^d} T_{\varepsilon,k}), \quad T_{\varepsilon,k} := \varepsilon k + \varepsilon^\alpha T, \alpha > 1.$$

With a proper scaling, the model in $(0,T) \times \Omega_\varepsilon$ reads:

$$\begin{cases} \varepsilon^\lambda(\partial_t \mathrm{u}_\varepsilon + (\mathrm{u}_\varepsilon \cdot \nabla)\mathrm{u}_\varepsilon) - \varepsilon^{\frac{3-\alpha}{2}} \mathrm{div}\{\eta_r(\varepsilon^{\frac{3-\alpha}{2}} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon\} + \nabla p_\varepsilon = f, \\ \mathrm{div}\, \mathrm{u}_\varepsilon = 0, \end{cases} \tag{6}$$

with

- non-slip boundary condition $\mathrm{u}_\varepsilon = 0$ on $(0,T) \times \partial\Omega_\varepsilon$.
- initial data $\mathrm{u}_\varepsilon|_{t=0} = \mathrm{u}_{0,\varepsilon} \in L^2(\Omega_\varepsilon)$.

Homogenization of non-Newtonian flows

Physical parameters (compressible analogy):

- Reynolds number: $\mathrm{Re} = \varepsilon^{\lambda+\alpha-3}$.
- Mach number: $\mathrm{Ma} = \varepsilon^{\frac{\lambda}{2}}$.
- Froude number: $\mathrm{Fr} = \varepsilon^{\frac{\lambda}{2}}$.
- Knudsen number: $\mathrm{Kn} \sim \mathrm{Ma}/\mathrm{Re} = \varepsilon^{3-\alpha-\frac{\lambda}{2}}$.
- Our setting: $1 < \alpha < \frac{3}{2}$, $\lambda > \alpha$.

It is reasonable to model the flow as a continuum if $\mathrm{Kn} \lesssim \varepsilon^{\alpha}$, which is the length scale of the holes and thus the smallest scale in the system.

In turn, the *physically* relevant values are $\lambda \leq 2(3-2\alpha)$. We can reach this physically relevant range as long as $\alpha < \frac{6}{5}$.

Main results

Theorem (L.-Oschmann 2024: Stationary case)

Let

$$1 < r < 3,\ 1 < \alpha < \frac{3}{2},\ \lambda > \alpha.$$

Let $(\mathrm{u}_\varepsilon, p_\varepsilon)$ be a finite energy stationary weak solution to (6). Then $p_\varepsilon = p_\varepsilon^{(1)} + \varepsilon^\sigma p_\varepsilon^{\mathrm{res}}$ with $\sigma > 0$ such that

$$\tilde{\mathrm{u}}_\varepsilon \to \mathrm{u} \text{ weakly in } L^2(\Omega),$$
$$\tilde{p}_\varepsilon^{(1)} \to p \text{ weakly in } L^2(\Omega),$$
$$\|\tilde{p}_\varepsilon^{\mathrm{res}}\|_{L^q(\Omega)} \le C \text{ for some } q > 1.$$

Moreover, the limit (u, p) satisfies the Darcy's law:

$$\begin{cases} \frac{1}{2}\eta_0 \mathrm{u} = M_0^{-1}(\mathrm{f} - \nabla p), & \text{in } \Omega, \\ \operatorname{div} \mathrm{u} = 0, & \text{in } \Omega, \\ \mathrm{u} \cdot \mathrm{n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

Main results

Theorem (L.-Oschmann 2024: Evolutionary case)

Let

$$1 < r < 3,\ 1 < \alpha < \frac{3}{2},\ \lambda > \alpha,\ \varepsilon^{\frac{\lambda}{2}} \|u_{\varepsilon 0}\|_{L^2(\Omega_\varepsilon)} \le C.$$

Let u_ε be a finite energy weak solution to (6) with initial datum $u_{\varepsilon 0}$. Then, there exists $P_\varepsilon = P_\varepsilon^{(1)} + P_\varepsilon^{\mathrm{res}}$ with

$$\|P_\varepsilon^{(1)}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \le C, \quad \|P_\varepsilon^{\mathrm{res}}\|_{L^\infty(0,T;L^q(\Omega_\varepsilon))} \le C\varepsilon^\sigma,\ q > 1, \sigma > 0,$$

such that $(u_\varepsilon, \partial_t P_\varepsilon)$ satisfies (6) in the sense of distribution. Moreover,

$$\tilde{u}_\varepsilon \to u \text{ weakly in } L^2((0,T)\times\Omega), \quad \tilde{P}_\varepsilon^{(1)} \to P \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),$$

where the limit (u, p) with $p = \partial_t P$, satisfies the Darcy's law:

$$\begin{cases} \frac{1}{2}\eta_0 u = M_0^{-1}(f - \nabla p),\ \operatorname{div} u = 0, & \text{in } (0,T)\times\Omega, \\ u \cdot n = 0, & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{8}$$

Main results

Theorem (L.-Oschmann 2024: Convergence rates)

Let
$$1 < r < 3,\ 1 < \alpha < \frac{3}{2},\ \lambda > \alpha,\ \varepsilon^{\frac{\lambda}{2}} \|\mathrm{u}_{\varepsilon 0}\|_{L^2(\Omega_\varepsilon)} \leq C.$$

Let u_ε be a weak solution to (6) with initial datum $\mathrm{u}_{\varepsilon 0} \in L^2(\Omega_\varepsilon)$.
Let $(\mathrm{u}, p) \in W^{1,\infty}(0, T; W^{2,\infty}(\Omega)) \times L^\infty(0, T; W^{1,\infty}(\Omega))$ be a strong solution to Darcy's law (8) with initial value $\|\mathrm{u}(0,\cdot)\|_{L^2(\Omega)} \leq C$. Then,

$$\|\tilde{\mathrm{u}}_\varepsilon - \mathrm{u}\|^2_{L^2((0,T)\times\Omega)} \leq C\Big(\varepsilon^\lambda \|\tilde{\mathrm{u}}_{\varepsilon 0} - \mathrm{u}(0,\cdot)\|^2_{L^2(\Omega)} + \varepsilon^{\alpha-1} + \varepsilon^{\lambda-\alpha-\frac{\theta}{3}(17-11\alpha)} + \varepsilon^{(3-2\alpha)|r-2|}\Big), \tag{9}$$

where $\theta \in (0,1)$ arbitrarily small. The last term in (9) can be taken to be zero if $r = 2$.

Estimates of u_ε

Poincaré inequality in porous media: $1 \leq q < 3, \alpha \geq 1$

$$\|\mathrm{u}\|_{L^q(\Omega_\varepsilon)} \leq C \min\left\{\varepsilon^{\frac{3-(3-q)\alpha}{q}}, 1\right\} \|\nabla \mathrm{u}\|_{L^q(\Omega_\varepsilon)}, \ \forall \ \mathrm{u} \in W_0^{1,q}(\Omega_\varepsilon).$$

Korn inequality: $1 < q < \infty$:

$$\|\nabla u\|_{L^q(\Omega_\varepsilon)} \leq C(q)\|Du\|_{L^q(\Omega_\varepsilon)}, \quad \text{for arbitrary } u \in W_0^{1,q}(\Omega_\varepsilon).$$

By energy inequality

$$\varepsilon^{3-\alpha} \int_{\Omega_\varepsilon} \eta_\infty |D\mathrm{u}_\varepsilon|^2 + (\eta_0 - \eta_\infty)(1 + \kappa|\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1} |D\mathrm{u}_\varepsilon|^2 \mathrm{d}x$$
$$\leq \int_{\Omega_\varepsilon} \mathrm{f} \cdot \mathrm{u}_\varepsilon \mathrm{d}x \leq C\varepsilon^{\frac{3-\alpha}{2}} \|D\mathrm{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\mathrm{f}\|_{L^2(\Omega_\varepsilon)}.$$

This implies

$$\varepsilon^{3-\alpha}\|D\mathrm{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{(3-\alpha)(r-1)}\|D\mathrm{u}_\varepsilon\|^r_{L^r(\Omega_\varepsilon)} \leq C.$$

Estimates of u_ε

Consequently,

$$\varepsilon^{\frac{3-\alpha}{2}}\|\nabla \mathrm{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \|\mathrm{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \varepsilon^{\frac{(3-\alpha)(r-1)}{r}}\|\nabla \mathrm{u}_\varepsilon\|_{L^r(\Omega_\varepsilon)} \leq C,$$

and, up to a subsequence,

$$\tilde{\mathrm{u}}_\varepsilon \to \mathrm{u} \text{ weakly in } L^2(\Omega), \quad \operatorname{div} \mathrm{u} = 0.$$

Estimates for p_ε

There exists a unique $p_\varepsilon \in L^q_0(\Omega_\varepsilon)$ for some $q > 1$, such that

$$\nabla p_\varepsilon = \varepsilon^{3-\alpha}\mathrm{div}\left(\eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon\right) - \varepsilon^{\lambda}\mathrm{div}\left(\mathrm{u}_\varepsilon \otimes \mathrm{u}_\varepsilon\right) + \mathrm{f}, \quad \text{in } \mathcal{D}'(\Omega_\varepsilon).$$

Bogovskii type operator in perforated domains (Diening-Feireisl-L. 2017):

There exists a linear operator

$$\mathcal{B}_\varepsilon : L^q_0(\Omega_\varepsilon) \to W^{1,q}_0(\Omega_\varepsilon; \mathbb{R}^3), \quad \text{for all } \ 1 < q < \infty,$$

such that for arbitrary $f \in L^q_0(\Omega_\varepsilon)$ there holds

$$\mathrm{div}\mathcal{B}_\varepsilon(f) = f \quad \text{a.e. in } \Omega_\varepsilon,$$

$$\|\mathcal{B}_\varepsilon(f)\|_{W^{1,q}_0(\Omega_\varepsilon)} \le C\left(1 + \varepsilon^{\frac{(3-q)\alpha-3}{q}}\right)\|f\|_{L^q(\Omega_\varepsilon)}.$$

Estimates for p_ε

Given any $\varphi \in C_c^\infty(\Omega_\varepsilon)$, define

$$\Phi = \mathcal{B}_\varepsilon(\varphi - \langle \varphi \rangle_{\Omega_\varepsilon}), \quad \langle \varphi \rangle_{\Omega_\varepsilon} := \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \varphi \, \mathrm{d}x.$$

Clearly $\Phi \in W_0^{1,q}(\Omega_\varepsilon; \mathbb{R}^3)$ for any $1 < q < \infty$ with estimates

$$\|\Phi\|_{W_0^{1,q}(\Omega_\varepsilon)} \leq C\left(1 + \varepsilon^{\frac{(3-q)\alpha-3}{q}}\right)\|\varphi\|_{L^q(\Omega_\varepsilon)}, \text{ for all } \ 1 < q < \infty.$$

The idea is to use Φ as a test function to derive the estimates of p_ε:

$$\begin{aligned}\langle p_\varepsilon, \varphi \rangle_{\Omega_\varepsilon} &= \langle p_\varepsilon, \varphi - \langle \varphi \rangle_{\Omega_\varepsilon} \rangle_{\Omega_\varepsilon} = \langle p_\varepsilon, \operatorname{div} \Phi \rangle_{\Omega_\varepsilon} = -\langle \nabla p_\varepsilon, \Phi \rangle_{\Omega_\varepsilon} \\ &= \varepsilon^{3-\alpha} \langle \eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon, \nabla \Phi \rangle_{\Omega_\varepsilon} - \varepsilon^\lambda \langle \mathrm{u}_\varepsilon \otimes \mathrm{u}_\varepsilon, \nabla \Phi \rangle_{\Omega_\varepsilon} - \langle \mathrm{f}, \Phi \rangle_{\Omega_\varepsilon}.\end{aligned}$$

Estimates of p_ε

Recall: $\eta_r(D\mathrm{u}_\varepsilon) = (\eta_0 - \eta_\infty)(1 + \kappa|D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1} + \eta_\infty$.

If $1 < r < 2$, $|\eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon)| \leq C$, then

$$\begin{aligned}
\varepsilon^{3-\alpha}|\langle \eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon, \nabla\Phi\rangle_{\Omega_\varepsilon}| &\leq C\varepsilon^{3-\alpha}\|D\mathrm{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\Phi\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{\frac{3-\alpha}{2}}\|\nabla\Phi\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\|\varphi\|_{L^2(\Omega_\varepsilon)},
\end{aligned}$$

where we used the quantitative estimates of Bogovskii type operator $\mathcal{B}_\varepsilon$:

$$\|\Phi\|_{W_0^{1,q}(\Omega_\varepsilon)} \leq C\left(1 + \varepsilon^{\frac{(3-q)\alpha-3}{q}}\right)\|\varphi\|_{L^q(\Omega_\varepsilon)}, \text{ for all } 1 < q < \infty.$$

Estimates of p_ε

If $2 < r < 3$,

$$|\eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon| \le C\varepsilon^{(3-\alpha)(r-2)} |D\mathrm{u}_\varepsilon|^{r-1} + C|D\mathrm{u}_\varepsilon|.$$

Then

$$\begin{aligned}
&\varepsilon^{3-\alpha} |\langle \eta_r(\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon) D\mathrm{u}_\varepsilon, \nabla\Phi\rangle_{\Omega_\varepsilon}| \\
&\le C\varepsilon^{(3-\alpha)(r-1)} \| |D\mathrm{u}_\varepsilon|^{r-1} \|_{L^{\frac{2}{r-1}}(\Omega_\varepsilon)} \|\nabla\Phi\|_{L^{\frac{2}{3-r}}(\Omega_\varepsilon)} \\
&\quad + C\varepsilon^{3-\alpha} \|D\mathrm{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla\Phi\|_{L^2(\Omega_\varepsilon)}. \\
&\le C\varepsilon^{\frac{(3-\alpha)(r-1)}{2}} \|\nabla\Phi\|_{L^{\frac{2}{3-r}}(\Omega_\varepsilon)} + C\varepsilon^{\frac{3-\alpha}{2}} \|\nabla\Phi\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{(3-2\alpha)(r-2)} \|\varphi\|_{L^{\frac{2}{3-r}}(\Omega_\varepsilon)} + C\|\varphi\|_{L^2(\Omega_\varepsilon)},
\end{aligned}$$

where the power $(3-2\alpha)(r-2) > 0$.

Estimates of p_ε

By interpolation,

$$\begin{aligned}\|u_\varepsilon \otimes u_\varepsilon\|_{L^{q_1}(\Omega_\varepsilon)} &\le \|u_\varepsilon \otimes u_\varepsilon\|_{L^1(\Omega_\varepsilon)}^{1-\theta}\|u_\varepsilon \otimes u_\varepsilon\|_{L^3(\Omega_\varepsilon)}^{\theta} \le C\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^{2(1-\theta)}\|u_\varepsilon\|_{L^6(\Omega_\varepsilon)}^{2\theta}\\ &\le C\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^{2(1-\theta)}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^{2\theta} \le C\varepsilon^{-\theta(3-\alpha)},\end{aligned}$$

where $\frac{1}{q_1} = (1-\theta) + \frac{\theta}{3} = 1 - \frac{2\theta}{3}$. Therefore,

$$\begin{aligned}\varepsilon^\lambda \left|\langle u_\varepsilon \otimes u_\varepsilon, \nabla\Phi\rangle_{\Omega_\varepsilon}\right| &\le \varepsilon^\lambda \|u_\varepsilon\|_{L^{2q_1}(\Omega_\varepsilon)}^2 \|\nabla\Phi\|_{L^{q_1'}(\Omega_\varepsilon)}\\ &\le C\varepsilon^{\lambda-\theta(3-\alpha)}\|\nabla\Phi\|_{L^{\frac{3}{2\theta}}(\Omega_\varepsilon)}\\ &\le C\varepsilon^{\lambda-\alpha-\theta(5-3\alpha)}\|\varphi\|_{L^{\frac{3}{2\theta}}(\Omega_\varepsilon)}.\end{aligned}$$

Since $\lambda > \alpha$ and $\alpha < \frac{3}{2}$, we can always choose $\theta > 0$ suitably small such that

$$\lambda - \alpha - \theta(5-3\alpha) > 0.$$

Estimates of p_ε

Finally, for $\theta > 0$ small,

$$|\langle p_\varepsilon, \varphi\rangle_{\Omega_\varepsilon}| \le C\|\varphi\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{\lambda-\alpha-\theta(5-3\alpha)}\|\varphi\|_{L^{\frac{3}{2\theta}}(\Omega_\varepsilon)}, \text{ if } 1 < r < 2,$$

$$|\langle p_\varepsilon, \varphi\rangle_{\Omega_\varepsilon}| \le C\|\varphi\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{\lambda-\alpha-\theta(5-3\alpha)}\|\varphi\|_{L^{\frac{3}{2\theta}}(\Omega_\varepsilon)} + C\varepsilon^{(3-2\alpha)(r-2)}\|\varphi\|_{L^{\frac{2}{3-r}}(\Omega_\varepsilon)}, \text{ if } 2 < r < 3.$$

This means,

$$p_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{\text{res}}, \quad p_\varepsilon^{\text{res}} = \varepsilon^{\lambda-\alpha-\theta(5-3\alpha)} p_\varepsilon^{(2)} + \varepsilon^{(3-2\alpha)(r-2)} p_\varepsilon^{(3)} 1_{r>2},$$

$$\|p_\varepsilon^{(1)}\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon^{(2)}\|_{L^{\frac{3}{3-2\theta}}(\Omega_\varepsilon)} + \|p_\varepsilon^{(3)}\|_{L^{\frac{2}{r-1}}(\Omega_\varepsilon)} \le C.$$

An issue on the test functions

To obtain the limit equations: passing to the limit in the weak formulation of the original equations.

However:

- The limit equations are defined in Ω: $C_c^\infty(\Omega)$ test functions.
- The original equations are defined in Ω_ε: $C_c^\infty(\Omega_\varepsilon)$ test functions.

$\varphi \in C_c^\infty(\Omega) \xrightarrow{\text{surgery}} \varphi_\varepsilon$ vanishing on the holes such that:

- φ_ε be a proper test function in Ω_ε,
- $\varphi_\varepsilon \to \varphi$.

Different ideas:

- L. Tartar: Cell problem.
- G. Allaire: Local problem.

Tatar's cell problem

Generalized cell problem (Tartar 1980, Jing 2020, L. 2020)

$$\begin{cases} -\Delta w^i + \nabla q^i = \eta^{\frac{1}{2}} e^i, & \text{in } Y_f := \left(-\frac{1}{2}, \frac{1}{2}\right)^d \setminus \eta T, \\ \operatorname{div} w^i = 0, & \text{in } Y_f, \\ w^i = 0, & \text{on } T, \\ (w^i, q^i) \text{ is } Y\text{-periodic.} \end{cases} \tag{10}$$

- $\eta = \varepsilon^{\alpha - 1}$
- $\{e^i\}_{i=1,\cdots,d}$ is the standard Euclidean coordinate of $\mathbb{R}^d$.
- The cell problem admits a unique weak solution $(w^i, q^i) \in W^{1,2}(Y_f) \times L^2_0(Y_f)$. Smooth.

Tartar's cell problem

Scale back and define

$$w^i_\varepsilon(\cdot) := w^i\left(\frac{\cdot}{\varepsilon}\right), \quad q^i_\varepsilon(\cdot) := q^i\left(\frac{\cdot}{\varepsilon}\right),$$

solving

$$\begin{cases} -\varepsilon^2 \Delta w^i_\varepsilon + \varepsilon \nabla q^i_\varepsilon = \eta^{\frac{1}{2}} e^i, & \text{in } \varepsilon Y_f, \\ \operatorname{div} w^i_\varepsilon = 0, & \text{in } \varepsilon Y_f, \\ w^i_\varepsilon = 0, & \text{on } \varepsilon T, \\ (w^i_\varepsilon, q^i_\varepsilon) \text{ is } \varepsilon Y\text{-periodic.} \end{cases}$$

Clearly w^i_ε vanishes on the holes in Ω_ε.

$\varphi \in C^\infty_c(\Omega) \rightsquigarrow w^i_\varepsilon \varphi$: a good test function in Ω_ε.

Drawback: no quantitative $W^{1,q}$ estimates for the cell problem.

Allaire's local problem

For general a_ε with $\eta := \frac{a_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon \to 0$, after a scaling a_ε^{-1} such that the size of the holes becomes $O(1)$, one obtains a domain of the type

$$\eta^{-1} Q_0 \setminus T \to \mathbb{R}^d \setminus T, \quad \text{as } \varepsilon \to 0.$$

Allaire employed the *local problem*:

$$\begin{cases} -\Delta v^i + \nabla p^i = 0, & \text{in } \mathbb{R}^d \setminus T, \\ \operatorname{div} v^i = 0, & \text{in } \mathbb{R}^d \setminus T, \\ v^i = 0, & \text{on } T, \\ v^i = e^i, & \text{at infinity.} \end{cases}$$

The permeability tensor M_0:

$$(M_0)_{i,j} = \int_{\mathbb{R}^d \setminus T} \nabla v^i : \nabla v^j \,\mathrm{d}x.$$

Allaire's local problem

The modification $(v_\varepsilon^i, p_\varepsilon^i)$ is defined as follows: in each εQ_k:

$$\begin{aligned}
&v_\varepsilon^i = e^i, \quad p_\varepsilon^i = 0, && \text{in } \varepsilon Q_k \setminus B(\varepsilon x_k, \tfrac{\varepsilon}{2}),\\
&-\Delta v_\varepsilon^i + \nabla p_\varepsilon^i = 0,\ \operatorname{div} v_\varepsilon^i = 0, && \text{in } B(\varepsilon x_k, \tfrac{\varepsilon}{2}) \setminus B(\varepsilon x_k, \tfrac{\varepsilon}{4}),\\
&v_\varepsilon^i(x) = v^i\Big(\frac{x-\varepsilon x_k}{a_\varepsilon}\Big),\ p_\varepsilon^i(x) = \frac{1}{a_\varepsilon} p^i\Big(\frac{x-\varepsilon x_k}{a_\varepsilon}\Big), && \text{in } B(\varepsilon x_k, \tfrac{\varepsilon}{4}) \setminus T_{\varepsilon,k},\\
&v_\varepsilon^i = 0, \quad p_\varepsilon^i = 0, && \text{in } T_{\varepsilon,k}.
\end{aligned}$$

Clearly v_ε^i vanishes on the holes in Ω_ε.

$\varphi \in C_c^\infty(\Omega) \rightsquigarrow v_\varepsilon^i \varphi$: a good test function in Ω_ε.

Moreover, quantitative $W^{1,q}$ estimates hold.

Estimates of the local problem

Allaire 1990, Höfer, Kowalczyk, and Schwarzacher 2021.

Let $1 < \alpha < 3$. Then

- $\|v_\varepsilon^i\|_{L^\infty(\Omega)} + \varepsilon^{\frac{3-\alpha}{2}}\left(\|\nabla v_\varepsilon^i\|_{L^2(\Omega)} + \|q_\varepsilon^i\|_{L^2(\Omega)}\right) \leq C$;
- $\operatorname{div} v_\varepsilon^i = 0$ in Ω, $v_\varepsilon^i = 0$ on the holes $T_{\varepsilon,k}$ for all k, and $v_\varepsilon^i \to e^i$ strongly in $L^2(\Omega)$;
- for any $q > \frac{3}{2}$, $\|\nabla v_\varepsilon^i\|_{L^q(\Omega)} + \|q_\varepsilon^i\|_{L^q(\Omega)} \leq C\varepsilon^{-\alpha+\frac{3(\alpha-1)}{q}}$.

Estimates of the local problem

Moreover,

- for any $\varphi \in C_c^\infty(\Omega)$, and for any family $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying $\gamma_\varepsilon = 0$ on the holes $T_{\varepsilon,k}$ for all k and

$$\gamma_\varepsilon \to \gamma \text{ weakly in } L^2(\Omega), \quad \varepsilon^{\frac{3-\alpha}{2}} \|\nabla \gamma_\varepsilon\|_{L^2(\Omega)} \le C,$$

there holds

$$\varepsilon^{3-\alpha} \langle -\Delta \mathrm{v}_\varepsilon^i + \nabla q_\varepsilon^i, \varphi \gamma_\varepsilon \rangle_\Omega \to \int_\Omega \varphi M_0 \mathrm{e}^i \cdot \gamma \mathrm{d}x,$$

where M_0 is the permeability tensor defined by

$$(M_0)_{i,j} = \int_{\mathbb{R}^3 \setminus T} \nabla \mathrm{v}^i : \nabla \mathrm{v}^j \mathrm{d}x.$$

Passing to the limit

Given any scalar function $\varphi \in C_c^\infty(\Omega)$, testing by $\varphi \mathrm{v}_\varepsilon^i$ implies

$$\int_{\Omega_\varepsilon} -\varepsilon^\lambda \mathrm{u}_\varepsilon \otimes \mathrm{u}_\varepsilon : \nabla(\varphi \mathrm{v}_\varepsilon^i) + \varepsilon^{3-\alpha}\eta_r(\varepsilon^{3-\alpha}D\mathrm{u}_\varepsilon)D\mathrm{u}_\varepsilon : D(\varphi \mathrm{v}_\varepsilon^i)\mathrm{d}x$$
$$= \int_{\Omega_\varepsilon} p_\varepsilon \mathrm{div}\,(\varphi \mathrm{v}_\varepsilon^i)\mathrm{d}x + \int_{\Omega_\varepsilon} \mathrm{f} \cdot \varphi \mathrm{v}_\varepsilon^i \mathrm{d}x.$$

After a long journey,

$$\frac{\eta_0}{2}\int_\Omega \varphi M_0 \mathrm{e}^i \cdot \mathrm{u}\mathrm{d}x - \int_\Omega p\mathrm{div}\,(\varphi \mathrm{e}^i)\mathrm{d}x = \int_\Omega \mathrm{f} \cdot \varphi \mathrm{e}^i \mathrm{d}x, \quad \text{for each } \mathrm{e}^i.$$

Since M_0 is positive definite, this is exactly the Darcy's law :

$$\frac{1}{2}\eta_0 \mathrm{u} = M_0^{-1}(\mathrm{f} - \nabla p), \text{ in } \Omega.$$

Passing to the limit

The term related to the nonlinear stress tensor:

$$\begin{aligned}\eta_r(D\mathrm{u}_\varepsilon) &= (\eta_0-\eta_\infty)(1+\kappa|D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1}+\eta_\infty \\ &= (\eta_0-\eta_\infty)[(1+\kappa|D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1}-1]+\eta_0.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\Omega_\varepsilon}\varepsilon^{3-\alpha}\eta_r(\varepsilon^{3-\alpha}D\mathrm{u}_\varepsilon)D\mathrm{u}_\varepsilon : D(\varphi \mathrm{v}^i_\varepsilon)\mathrm{d}x \\ &= \int_{\Omega_\varepsilon}\varepsilon^{3-\alpha}(\eta_0-\eta_\infty)[(1+\kappa|D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1}-1]D\mathrm{u}_\varepsilon : D(\varphi \mathrm{v}^i_\varepsilon)\mathrm{d}x \\ &\quad + \int_{\Omega_\varepsilon}\varepsilon^{3-\alpha}\eta_0 D\mathrm{u}_\varepsilon : D(\varphi \mathrm{v}^i_\varepsilon)\mathrm{d}x.\end{aligned}$$

For example:

For $2 < r < 3$, using $0 \leq (1+s)^\alpha - 1 \leq s^\alpha$ for any $s \geq 0$ and $0 < \alpha \leq 1$,

$$\left|(1+\kappa|\varepsilon^{3-\alpha} D\tilde{\mathrm{u}}_\varepsilon|^2)^{\frac{r}{2}-1} - 1\right| \leq C\varepsilon^{(3-\alpha)(r-2)}|D\tilde{\mathrm{u}}_\varepsilon|^{r-2}.$$

Then,

$$\begin{aligned}
&\varepsilon^{3-\alpha}\left|\int_\Omega \left((1+\kappa|\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1} - 1\right) D\tilde{\mathrm{u}}_\varepsilon : D(\varphi \mathrm{v}_\varepsilon^i)\mathrm{d}x\right| \\
&\leq C\varepsilon^{(3-\alpha)(r-1)} \int_\Omega |D\tilde{\mathrm{u}}_\varepsilon|^{r-1} : |D(\varphi \mathrm{v}_\varepsilon^i)|\mathrm{d}x \\
&\leq C\varepsilon^{(3-\alpha)(r-1)} \| |\nabla \tilde{\mathrm{u}}_\varepsilon|^{r-1} \|_{L^{\frac{r}{r-1}}(\Omega)} \|\mathrm{v}_\varepsilon^i\|_{W^{1,r}(\Omega)} \\
&\leq C\varepsilon^{(3-\alpha)(r-1)} \varepsilon^{-(3-\alpha)\frac{(r-1)^2}{r}} \varepsilon^{-\alpha+\frac{3(\alpha-1)}{r}} = C\varepsilon^{\frac{(3-2\alpha)(r-2)}{r}} \to 0,
\end{aligned} \tag{11}$$

under the assumption $1 < \alpha < \frac{3}{2}$. Here we also used the fact $\frac{r}{r-1} > \frac{3}{2}$.

Evolutionary case

Integrating in time variable by introducing

$$\mathrm{U}_\varepsilon = \int_0^t \mathrm{u}_\varepsilon \,\mathrm{d}s,\;\; \mathrm{G}_\varepsilon = \int_0^t (\mathrm{u}_\varepsilon \otimes \mathrm{u}_\varepsilon)\,\mathrm{d}s,$$

$$\mathrm{H}_\varepsilon = \int_0^t (1+\kappa|\varepsilon^{3-\alpha} D\mathrm{u}_\varepsilon|^2)^{\frac{r}{2}-1} D\mathrm{u}_\varepsilon \,\mathrm{d}s,\;\; \mathrm{F} = \int_0^t \mathrm{f}\,\mathrm{d}s.$$

The classical theory of Stokes equations ensures the existence of

$$P_\varepsilon \in \begin{cases} C_{\rm weak}([0,T]; L^2_0(\Omega_\varepsilon)), & \text{if } 1 < r \le 2,\\ C_{\rm weak}([0,T]; L_0^{\frac{2}{r-1}}(\Omega_\varepsilon)), & \text{if } 2 \le r < 3, \end{cases}$$

such that for each $t \in [0,T]$,

$$\nabla P_\varepsilon = \mathrm{F} - \varepsilon^\lambda(\mathrm{u}_\varepsilon - \mathrm{u}_{\varepsilon 0}) - \varepsilon^\lambda \mathrm{div}\, \mathrm{G}_\varepsilon + \varepsilon^{3-\alpha}\frac{\eta_\infty}{2}\Delta \mathrm{U}_\varepsilon + \varepsilon^{3-\alpha}(\eta_0 - \eta_\infty)\mathrm{div}\, \mathrm{H}_\varepsilon \text{ in } \mathcal{D}'(\Omega_\varepsilon).$$

Convergence rates

Relative energy

$$E_\varepsilon(\mathrm{u}_\varepsilon|\mathrm{U}) = \frac{1}{2}\varepsilon^\lambda|\mathrm{u}_\varepsilon - \mathrm{U}|^2, \quad \forall\, \mathrm{U} \in C^\infty([0,T]\times\Omega;\mathbb{R}^3), \quad \mathrm{U}|_{\partial\Omega_\varepsilon} = 0.$$

Relative energy inequality

$$\left[\int_{\Omega_\varepsilon} E_\varepsilon(\mathrm{u}_\varepsilon|\mathrm{U})\mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau\int_{\Omega_\varepsilon} \varepsilon^{3-\alpha}[\eta_r(\varepsilon^{3-\alpha}D\mathrm{u}_\varepsilon)D\mathrm{u}_\varepsilon - \eta_r(\varepsilon^{3-\alpha}D\mathrm{U})D\mathrm{U}] : D(\mathrm{u}_\varepsilon - \mathrm{U})\mathrm{d}x\mathrm{d}t$$
$$\leq \int_0^\tau\int_{\Omega_\varepsilon} \varepsilon^{3-\alpha}\eta_r(\varepsilon^{3-\alpha}D\mathrm{U})D\mathrm{U} : (D\mathrm{U} - D\mathrm{u}_\varepsilon) - \varepsilon^\lambda\int_0^\tau\int_{\Omega_\varepsilon}(\mathrm{u}_\varepsilon - \mathrm{U})\cdot(\partial_t\mathrm{U} + (\mathrm{u}_\varepsilon\cdot\nabla)\mathrm{U})\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau\int_{\Omega_\varepsilon} \mathrm{f}\cdot(\mathrm{u}_\varepsilon - \mathrm{U})\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\Omega_\varepsilon} P_\varepsilon \operatorname{div}\partial_t\mathrm{U}\mathrm{d}x\mathrm{d}t - \int_{\Omega_\varepsilon}(P_\varepsilon(\tau,\cdot)\operatorname{div}\mathrm{U}(\tau,\cdot)\mathrm{d}x.$$

Denote $W_\varepsilon = (\mathrm{v}_\varepsilon^1, \mathrm{v}_\varepsilon^2, \mathrm{v}_\varepsilon^3)$ and $Q_\varepsilon = (q_\varepsilon^1, q_\varepsilon^2, q_\varepsilon^3)^T$. Then

$$\|W_\varepsilon - \mathbb{I}\|_{L^q(\Omega)} \leq C\varepsilon^{\min\{1,\frac{3}{q}\}(\alpha-1)} \quad \forall q \in [2,\infty).$$

Let u be a regular strong solution of Darcy's law. Choose $\mathrm{U} = W_\varepsilon \mathrm{u}$.

Thank you very much!