Random Compressible Fluid Flows

Mária Lukáčová
University of Mainz, Germany

E. Feireisl (Prague), B. She (CNU, Beijing), Y. Yuan (NUAA, Nanjing)

September 2024 MPDE24, Prague

Meteorology: Navier-Stokes-cloud model

$$
\begin{aligned}
\partial_t \rho' + \nabla \cdot (\rho \boldsymbol{u}) &= 0 \\
\partial_t (\rho \boldsymbol{u}) + \nabla \cdot \left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p' \mathsf{Id} - \rho \mu_m \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) \right) &= -\rho' g \boldsymbol{e_3} \\
\partial_t (\rho \, \theta)' + \nabla \cdot (\rho \theta \boldsymbol{u} - \rho \mu_h \nabla \theta) &= S_\theta(C, E, \rho, \theta)
\end{aligned}
$$

Meteorology: Navier-Stokes-cloud model

$$\begin{aligned}
\partial_t \rho' + \nabla\cdot(\rho\boldsymbol{u}) &= 0\\
\partial_t(\rho\boldsymbol{u}) + \nabla\cdot\left(\rho\boldsymbol{u}\otimes\boldsymbol{u} + p'\mathsf{Id} - \rho\mu_m\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)\right) &= -\rho' g\boldsymbol{e_3}\\
\partial_t(\rho\,\theta)' + \nabla\cdot(\rho\theta\boldsymbol{u} - \rho\mu_h\nabla\theta) &= S_\theta(C,E,\rho,\theta)
\end{aligned}$$

$$\begin{aligned}
\partial_t(\rho q_v) + \nabla\cdot(\rho q_v\boldsymbol{u} - \mu_h\rho\nabla q_v) &= \rho(-C+E)\\
\partial_t(\rho q_c) + \nabla\cdot(\rho q_c\boldsymbol{u} - \mu_h\rho\nabla q_c) &= \rho(C - A_1 - A_2)\\
\partial_t(\rho q_r) + \nabla\cdot(-v_q\rho q_r\boldsymbol{e_3} + \rho q_r\boldsymbol{u} - \mu_h\rho\nabla q_r) &= \rho(A_1 + A_2 - E)
\end{aligned}$$

Meteorology: Navier-Stokes-cloud model

$$
\begin{aligned}
\partial_t \rho' + \nabla \cdot (\rho \boldsymbol{u}) &= 0 \\
\partial_t (\rho \boldsymbol{u}) + \nabla \cdot \left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p' \mathsf{Id} - \rho \mu_m \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) \right) &= -\rho' g \boldsymbol{e_3} \\
\partial_t (\rho\, \theta)' + \nabla \cdot (\rho \theta \boldsymbol{u} - \rho \mu_h \nabla \theta) &= S_\theta(C, E, \rho, \theta)
\end{aligned}
$$

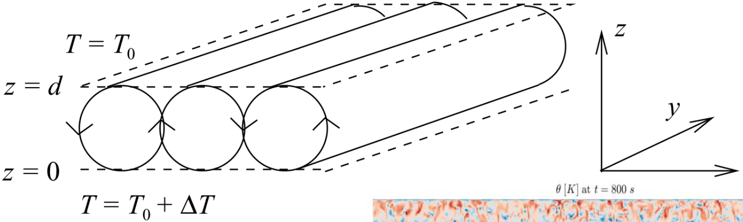
$$
\begin{aligned}
\partial_t(\rho q_v) + \nabla \cdot (\rho q_v \boldsymbol{u} - \mu_h \rho \nabla q_v) &= \rho(-C + E) \\
\partial_t(\rho q_c) + \nabla \cdot (\rho q_c \boldsymbol{u} - \mu_h \rho \nabla q_c) &= \rho(C - A_1 - A_2) \\
\partial_t(\rho q_r) + \nabla \cdot (-v_q \rho q_r \boldsymbol{e_3} + \rho q_r \boldsymbol{u} - \mu_h \rho \nabla q_r) &= \rho(A_1 + A_2 - E)
\end{aligned}
$$

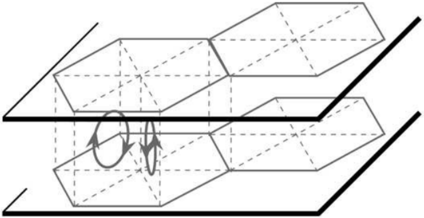
• to describe *uncertainty propagation of data* $\Longrightarrow$ the solution is a random process

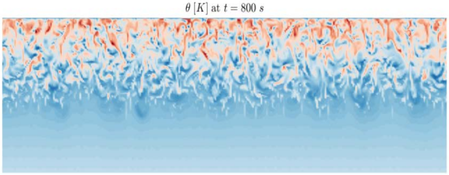
• Chertock, Kurganov, ML, Spichtinger, Wiebe: Math. Clim. Weather Forecast ('19), JCP ('23)

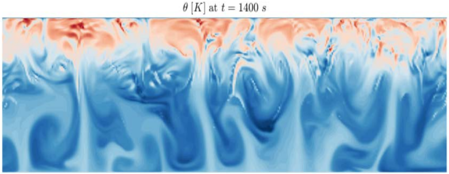
-second order FVM in space-time & spectral approx. in random space

Test: Rayleigh Bénard Convection



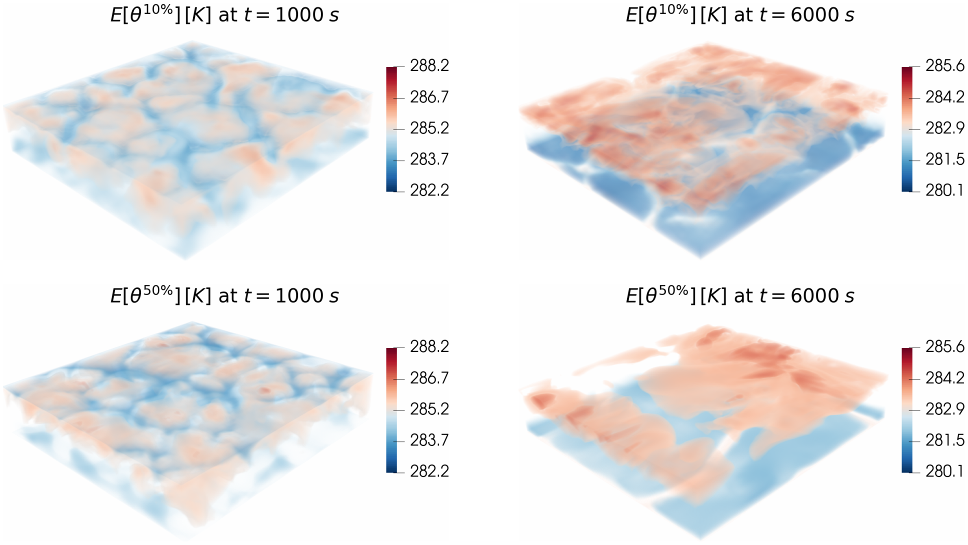






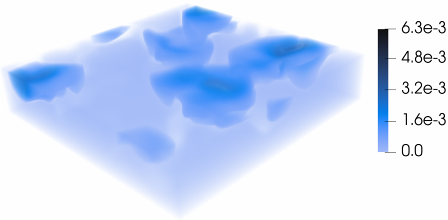
3D Rayleigh Bénard Convection

Time evolution of the potential temperature in 3D

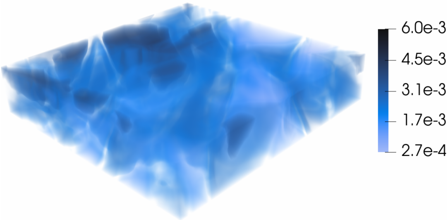


Time evolution of rain: Mean value

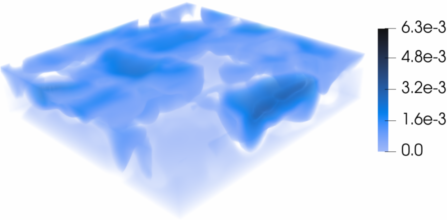
$E[q_r^{10\%}]$ at $t = 1000\ s$



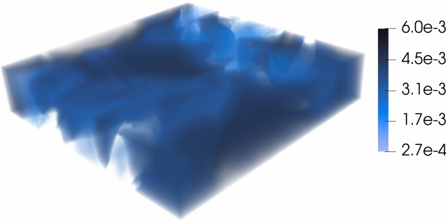
$E[q_r^{10\%}]$ at $t = 6000\ s$



$E[q_r^{50\%}]$ at $t = 1000\ s$



$E[q_r^{50\%}]$ at $t = 6000\ s$



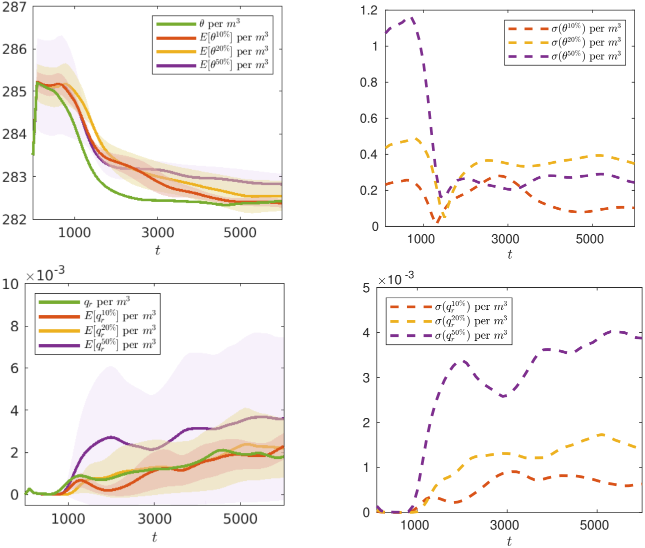


Figure: Time evolution of the expected values (left) with their standard deviations (right) for the potential temperature θ and mass concentration of rain q_r per m^3

Shallow water system with bottom topgraphy

• Chertock, Herty, Ishakov, Janajra, Kurganov, ML: Comm.Appl.Math.Comp. ('24)

- FVM with 2nd order in space-time & 5th order Ai-WENO-Z in random space
- random water surface, discharge & deterministic bottom topography

Shallow water system with bottom topgraphy

- Chertock, Herty, Ishakov, Janajra, Kurganov, ML: Comm.Appl.Math.Comp. ('24)

- FVM with 2nd order in space-time & 5th order Ai-WENO-Z in random space
- random water surface, discharge & deterministic bottom topography

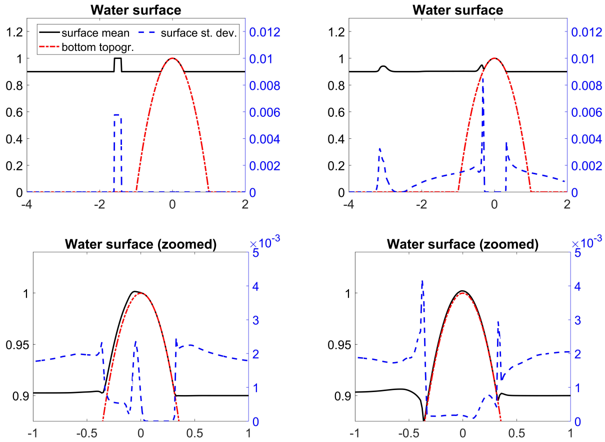


Figure: Mean and standard deviation of water surface at $t = 0$ (top left), 0.5 (top right), 0.75 (bottom left), and 1 (bottom right)

Random compressible flows

- compressible Navier-Stokes-Fourier / Euler equations:
- conservation of mass, momentum, energy
- second law of thermodynamics

Random compressible flows

- compressible Navier-Stokes-Fourier / Euler equations:
- conservation of mass, momentum, energy
- second law of thermodynamics
- Goal: rigorous convergence analysis of Monte Carlo FVM, stochastic collocation FVM without assuming existence of regular exact sol.
  uncertainties in initial/boundary data, model parameters

Random compressible flows

- compressible Navier-Stokes-Fourier / Euler equations:
- conservation of mass, momentum, energy
- second law of thermodynamics
- Goal: rigorous convergence analysis of Monte Carlo FVM, stochastic collocation FVM without assuming existence of regular exact sol. uncertainties in initial/boundary data, model parameters
- rigorous analysis for random elliptic, parabolic problems: Babuška et al. ('10), Nobile et al. ('08), Charrier et al. ('13), Bespalov et al. ('12), Mishra & Schwab ('12), Cohen & Schwab ('11), ...

Random compressible flows

- compressible Navier-Stokes-Fourier / Euler equations:
- conservation of mass, momentum, energy
- second law of thermodynamics
- Goal: rigorous convergence analysis of Monte Carlo FVM, stochastic collocation FVM without assuming existence of regular exact sol.
  uncertainties in initial/boundary data, model parameters
- rigorous analysis for random elliptic, parabolic problems: Babuška et al. ('10), Nobile et al. ('08), Charrier et al. ('13), Bespalov et al. ('12), Mishra & Schwab ('12), Cohen & Schwab ('11), ...
- **fundamental problem:** weak solutions of compressible flows are in general not unique
  pathwise arguments for statistical convergence analysis do not work!

Barotropic Navier-Stokes equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0$$
$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\mu, \lambda, \nabla_x \boldsymbol{u})$$

$$\mathbb{S}(\mu, \lambda, \nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u} - \frac{2}{d} \mathrm{div}_x \boldsymbol{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I}, \; d = 2, 3$$
$$p(\varrho) = a \varrho^\gamma, \quad a > 0, \quad \gamma > 1$$

- total energy

$$E = \frac{1}{2} \varrho \boldsymbol{u}^2 + \frac{1}{\gamma - 1} p(\varrho)$$

plays the role of entropy

$$\frac{d}{dt} \int_\Omega E(t) \, \mathrm{d}x \leq 0$$

-periodic BC

Known results ...

- local well-posedness (regularity) Valli-Zajaczkowski ('86), Cho-Kim ('06), Kotschote ('15)
- global weak solutions Lions ('98) for $\gamma > 9/5$, Feireisl-Novotný-Petzeltová ('01) for $\gamma > d/2$
- Buckmaster et al. ('22), Merle et al. ('22) originally regular sol. may develop finite time blow up

Main Goal:

statistical analysis of random compressible NS eqs.
by means of the Monte Carlo/stochastic collocation FVM method

Main Goal:
statistical analysis of random compressible NS eqs.
by means of the Monte Carlo/stochastic collocation FVM method

I. Deterministic convergence/error analysis

Lax equivalence principle:
stability + consistency $\Leftrightarrow$ convergence

Finite volume scheme

- Q_h p.w. constant functions on regular rectangular grid $\mathcal{T}_h$

$$\Pi : L^1(\Omega) \to Q_h. \quad \Pi\phi = \sum_{K \in \mathcal{T}_h} 1_K \frac{1}{|K|} \int_K \phi \, \mathrm{d}x$$

Finite volume scheme

- Q_h p.w. constant functions on regular rectangular grid $\mathcal{T}_h$

$$\Pi: L^1(\Omega) \to Q_h. \quad \Pi\phi = \sum_{K \in \mathcal{T}_h} 1_K \frac{1}{|K|} \int_K \phi \, \mathrm{d}x$$

- average and jump on edge $\sigma = K|L$

$$\{\!\{v\}\!\}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \; [\![v]\!] = v^{\text{out}}(x) - v^{\text{in}}(x)$$

- upwind velocity

$$r^{\text{up}} = \begin{cases} r^{\text{in}} & \text{if } \{\!\{\boldsymbol{v}\}\!\} \cdot \boldsymbol{n} \geq 0, \\ r^{\text{out}} & \text{if } \{\!\{\boldsymbol{v}\}\!\} \cdot \boldsymbol{n} < 0. \end{cases}$$

$$\mathrm{Up}\,[r, \boldsymbol{v}] = r^{\text{up}} \boldsymbol{v} \cdot \boldsymbol{n}$$

Finite volume scheme

- Q_h p.w. constant functions on regular rectangular grid $\mathcal{T}_h$

$$\Pi : L^1(\Omega) \to Q_h. \quad \Pi\phi = \sum_{K \in \mathcal{T}_h} 1_K \frac{1}{|K|} \int_K \phi \, \mathrm{d}x$$

- average and jump on edge $\sigma = K|L$

$$\{\!\{v\}\!\}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \ [\![v]\!] = v^{\text{out}}(x) - v^{\text{in}}(x)$$

- upwind velocity

$$r^{\text{up}} = \begin{cases} r^{\text{in}} & \text{if } \{\!\{\boldsymbol{v}\}\!\} \cdot \boldsymbol{n} \geq 0, \\ r^{\text{out}} & \text{if } \{\!\{\boldsymbol{v}\}\!\} \cdot \boldsymbol{n} < 0. \end{cases}$$

$$\mathrm{Up}\,[r, \boldsymbol{v}] = r^{\text{up}} \boldsymbol{v} \cdot \boldsymbol{n}$$

- **numerical flux**

$$F_h(r, \boldsymbol{v}) = \mathrm{Up}\,[r, \boldsymbol{v}] - h^{\varepsilon} [\![r]\!], \quad \varepsilon > 0$$

- discrete divergence

$$\mathrm{div}_h \boldsymbol{u}_h(\mathbf{x}) := \sum_{K \in \mathcal{T}_h} (\mathrm{div}_h \boldsymbol{u}_h)_K 1_K, \quad (\mathrm{div}_h \boldsymbol{u}_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \left\{\!\!\left\{ \boldsymbol{u}_h \right\}\!\!\right\} \cdot \boldsymbol{n}$$

- discrete divergence

$$\mathrm{div}_h \boldsymbol{u}_h(\mathbf{x}) := \sum_{K \in \mathcal{T}_h} (\mathrm{div}_h \boldsymbol{u}_h)_K 1_K, \quad (\mathrm{div}_h \boldsymbol{u}_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \{\!\{ \boldsymbol{u}_h \}\!\} \cdot \boldsymbol{n}$$

- **Finite Volume Method**

$$\int_\Omega D_t \varrho_h^k \varphi_h \mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h(\varrho_h^k, \boldsymbol{u}_h^k) [\![\varphi_h]\!] dS_x = 0 \quad \text{for all } \varphi_h \in Q_h$$

- discrete divergence

$$\mathrm{div}_h \boldsymbol{u}_h(\mathbf{x}) := \sum_{K \in \mathcal{T}_h} (\mathrm{div}_h \boldsymbol{u}_h)_K 1_K, \quad (\mathrm{div}_h \boldsymbol{u}_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}$$

- **Finite Volume Method**

$$\int_\Omega D_t \varrho_h^k \varphi_h \mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h(\varrho_h^k, \boldsymbol{u}_h^k) [\![\varphi_h]\!] dS_x = 0 \quad \text{for all } \varphi_h \in Q_h$$

$$\int_\Omega D_t(\varrho_h^k \boldsymbol{u}_h^k) \cdot \boldsymbol{\varphi}_h \mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h(\varrho_h^k \boldsymbol{u}_h^k, \boldsymbol{u}_h^k) \cdot [\![\boldsymbol{\varphi}_h]\!] - \left\{\!\!\left\{ p_h^k \right\}\!\!\right\} \boldsymbol{n} \cdot [\![\boldsymbol{\varphi}_h]\!] dS_x$$

$$= -\mu \sum_{\sigma \in \mathcal{E}} \int_\sigma \frac{1}{h} [\![\boldsymbol{u}_h^k]\!] \cdot [\![\boldsymbol{\varphi}_h]\!] dS_x - (\mu + \lambda) \int_\Omega \mathrm{div}_h \boldsymbol{u}_h^k \, \mathrm{div}_h \boldsymbol{\varphi}_h \mathrm{d}x$$

$$\text{for all } \boldsymbol{\varphi}_h \in \boldsymbol{Q}_h$$

Discrete energy dissipation: Stability

$$D_t \int_\Omega \left(\frac{1}{2} \varrho_h^k |\boldsymbol{u}_h^k|^2 + \mathcal{P}(\varrho_h^k) \right) \, \mathrm{d}x +$$

$$h^\varepsilon \sum_{\sigma \in \mathcal{E}} \int_\sigma \{\!\{ \varrho_h^k \}\!\} [\![\boldsymbol{u}_h^k]\!]^2 dS_x + \mu \| \nabla_{\mathcal{E}} \boldsymbol{u}_h^k \|_{L^2}^2 + (\mu + \lambda) \int_\Omega |\mathrm{div}_h \boldsymbol{u}_h^k|^2 \, \mathrm{d}x$$

$$= -\frac{\Delta t}{2} \int_\Omega \mathcal{P}''(\xi) |D_t \varrho_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathcal{P}''(\zeta) [\![\varrho_h^k]\!]^2 \left(h^\varepsilon + |\{\!\{ \boldsymbol{u}_h^k \}\!\} \cdot \boldsymbol{n}| \right) dS_x$$

$$- \frac{\Delta t}{2} \int_\Omega \varrho_h^{k-1} |D_t \boldsymbol{u}_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\{\!\{ \boldsymbol{u}_h^k \}\!\} \cdot \boldsymbol{n}| [\![\boldsymbol{u}_h^k]\!]^2 dS_x$$

$$\mathcal{P}(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma \ldots \text{ pressure potential}$$

Consistency

Let $\Delta t \approx h$, if $1 < \gamma < 2$ then $-1 < \varepsilon < 2\gamma - 1 - d/3$, else $\varepsilon > -1$

$$\left[\int_\Omega \varrho_h(\tau)\phi(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{t_k} = \int_0^{t_k}\int_\Omega \left[\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \phi\right]\ \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^{t_k} e_{1,h}(t_k,\phi)\mathrm{d}t$$

for any $\phi \in C^2([0,T]\times\Omega)$, $k = 0,\dots,N$, $N\Delta t = T$

Consistency

Let $\Delta t \approx h$, if $1 < \gamma < 2$ then $-1 < \varepsilon < 2\gamma - 1 - d/3$, else $\varepsilon > -1$

$$\left[\int_\Omega \varrho_h(\tau)\phi(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{t_k} = \int_0^{t_k}\int_\Omega \left[\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \phi\right]\ \mathrm{d}x\mathrm{d}t$$
$$+\int_0^{t_k} e_{1,h}(t_k,\phi)\mathrm{d}t$$

for any $\phi \in C^2([0,T]\times\Omega)$, $k = 0,\dots,N$, $N\Delta t = T$

$$\left[\int_\Omega \varrho_h(\tau)\boldsymbol{u}_h(\tau)\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{t_k} =$$
$$\int_0^{t_k}\int_\Omega \left[\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + + p_h \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t,$$
$$-\int_0^{t_k}\int_\Omega \mu \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + (\mu+\lambda)\mathrm{div}_h \boldsymbol{u}_h\, \mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^{t_k} e_{2,h}(t,\boldsymbol{\varphi})\mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C^2([0,T]\times\Omega)$, $k = 0,\dots N$, $N\Delta t = T$

Consistency

Let $\Delta t \approx h$, if $1 < \gamma < 2$ then $-1 < \varepsilon < 2\gamma - 1 - d/3$, else $\varepsilon > -1$

$$\left[\int_\Omega \varrho_h(\tau)\phi(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{t_k} = \int_0^{t_k}\int_\Omega \left[\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \phi\right]\ \mathrm{d}x\mathrm{d}t$$
$$+\int_0^{t_k} e_{1,h}(t_k,\phi)\mathrm{d}t$$

for any $\phi \in C^2([0,T]\times\Omega)$, $k = 0,\dots,N$, $N\Delta t = T$

$$\left[\int_\Omega \varrho_h(\tau)\boldsymbol{u}_h(\tau)\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{t_k} =$$
$$\int_0^{t_k}\int_\Omega \left[\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + + p_h \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t,$$
$$-\int_0^{t_k}\int_\Omega \mu \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + (\mu+\lambda)\mathrm{div}_h \boldsymbol{u}_h\, \mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^{t_k} e_{2,h}(t,\boldsymbol{\varphi})\mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C^2([0,T]\times\Omega)$, $k = 0,\dots N$, $N\Delta t = T$

$$\|e_{j,h}(\cdot,\phi)\|_{L^1(0,T)} \lesssim h^\beta \|\phi\|_{C^2},\ j = 1,2,\ \text{for some}\ \ \beta > 0$$

Passing with $h \to 0$: convergence

• we do not have enough regularity from a priori estimates to pass to the limit as $h \to 0$ in the momentum upwind term

Passing with $h \to 0$: convergence

• we do not have enough regularity from a priori estimates to pass to the limit as $h \to 0$ in the momentum upwind term

• **BUT:** any sequence bdd. in L^p has a weakly convergent subsequence

Passing with $h \to 0$: convergence

• we do not have enough regularity from a priori estimates to pass to the limit as $h \to 0$ in the momentum upwind term

• **BUT:** any sequence bdd. in L^p has a weakly convergent subsequence

$\Longrightarrow$ **Dissipative weak solutions**

Dissipative weak solutions

A pair $(\varrho, \boldsymbol{u})$ such that $\varrho \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega))$,
$\boldsymbol{m} = \varrho \boldsymbol{u} \in C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, $\boldsymbol{u} \in L^2((0,T); W^{1,2}(\Omega))$

- **continuity eq.**

$$\left[\int_{\Omega} \varrho \phi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega} \varrho \partial_t \phi + \varrho \boldsymbol{u} \cdot \nabla_x \phi \,\mathrm{d}x\mathrm{d}t$$

  $\tau \in [0,T]$ and $\phi \in C^1\big([0,T] \times \Omega\big)$

Dissipative weak solutions

A pair $(\varrho, \boldsymbol{u})$ such that $\varrho \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega))$,
$\boldsymbol{m} = \varrho\boldsymbol{u} \in C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, $\boldsymbol{u} \in L^2((0,T); W^{1,2}(\Omega))$

- **continuity eq.**

$$\left[\int_\Omega \varrho\phi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \varrho\partial_t\phi + \varrho\boldsymbol{u}\cdot\nabla_x\phi\,\mathrm{d}x\mathrm{d}t$$

$\tau \in [0,T]$ and $\phi \in C^1\big([0,T]\times\Omega\big)$

- **momentum eq.**

$$\left[\int_\Omega \varrho\boldsymbol{u}\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} =$$
$$\int_0^\tau \int_\Omega \varrho\boldsymbol{u}\partial_t\boldsymbol{\varphi} + [\varrho\boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)Id] : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$
$$-\int_0^\tau \int_\Omega \mathbb{S}(\mathbf{D}(\boldsymbol{u})) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t)\mathrm{d}t,$$

$\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{sym}))$ ... Reynolds turbulent defect,
$\boldsymbol{\varphi} \in C^1\big([0,T]\times\Omega\big)$

- **energy inequality**

$$\int_\Omega \left(\frac{1}{2}\varrho \boldsymbol{u}^2 + \mathcal{P}(\varrho)\right)(\tau)\,\mathrm{d}x + \int_0^\tau \int_\Omega \mathcal{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$
$$+ \quad \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \le \int_\Omega \left(\frac{1}{2}\varrho \boldsymbol{u}^2 + \mathcal{P}(\varrho)\right)(0)\,\mathrm{d}x$$

  - energy defect $\mathfrak{E} \in L^\infty((0,T),\ \mathcal{M}^+(\overline{\Omega}))$

- **defect compatibility**

$$\underline{d}\ \mathfrak{E} \le \mathrm{tr}[\mathfrak{R}] \le \overline{d}\ \mathfrak{E} \qquad 0 < \underline{d} \le \overline{d}$$

Convergence of FVM

Theorem (Feireisl, ML, Mizerová, She (M^{2}AN'21))

Let $\{(\varrho_h, \boldsymbol{u}_h)\}_{h\searrow 0}$ *be upwind FV solutions,*
if $1 < \gamma < 2$ *then* $-1 < \varepsilon < 2\gamma - 1 - d/3$*, else* $\varepsilon > -1$.
Let

$$\varrho_0 > 0, \; \int_\Omega E(\rho_0, \boldsymbol{m}_0) \lesssim 1.$$

Then there is a subsequence of $\{\varrho_h, \boldsymbol{m}_h\}_{h\searrow 0}$ *that converges to a dissipative weak sol.*

$$\begin{aligned}
&\varrho_{h_n} \to \varrho \quad \text{weakly-(*) in } \; L^\infty((0,T); L^\gamma(\Omega)),\\
&\boldsymbol{m}_{h_n} \to \boldsymbol{m} \quad \text{weakly-(*) in } \; L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)),\\
&E_{h_n} \to \overline{E} \quad \text{weakly-(*) in } \; L^\infty(0,T; \mathcal{M}^+(\overline{\Omega})) \quad \text{as } h \to 0.
\end{aligned}$$

Nash hypothesis

- Do we have global strong sol.?
- Nash (1958)
  "Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density."

Nash hypothesis

- Do we have global strong sol.?
- Nash (1958)
  "Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density."
- global in time boundedness of certain variables $\Longrightarrow$ global in time strong sol.

Nash hypothesis

- Do we have global strong sol.?
- Nash (1958)
  "Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density."
- global in time boundedness of certain variables $\Longrightarrow$ global in time strong sol.
- Feireisl, Wen, Zhu (2022): Navier-Stokes-Fourier: **viscous heat conductive fluids**:
  if $\|\varrho\|_{L^\infty((0,T)\times\Omega)} \lesssim 1,\ \|\vartheta\|_{L^\infty((0,T)\times\Omega)} \lesssim 1$ then local (in time) strong sol. remains strong sol. on $(0,T)$

Nash hypothesis

- Do we have global strong sol.?
- Nash (1958)
  "Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density."
- global in time boundedness of certain variables $\Longrightarrow$ global in time strong sol.
- Feireisl, Wen, Zhu (2022): Navier-Stokes-Fourier: **viscous heat conductive fluids**:
  if $\|\varrho\|_{L^\infty((0,T)\times\Omega)} \lesssim 1,\ \|\vartheta\|_{L^\infty((0,T)\times\Omega)} \lesssim 1$ then local (in time) strong sol. remains strong sol. on $(0,T)$
- Sun, Wang, Zhang (2011): **viscous, barotropic fluids**:
  if $\|\varrho\|_{L^\infty((0,T)\times\Omega)} \lesssim 1,\ \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \lesssim 1$ then local (in time) strong sol. remains strong sol. on $(0,T)$

Bounded numerical solutions

Theorem (Feireisl, ML, Mizerová, She (M2AN '21))

Let $\varrho_0 \in W^{3,2}(\Omega),\ 0 < \underline{\varrho} \leq \min \varrho_0,\ \boldsymbol{m}_0 \in W^{3,2}(\Omega)$.
and

$$\|(\varrho_h, \boldsymbol{u}_h)\|_{L^\infty((0,T)\times\Omega)} \leq c \qquad \textit{uniformly for } h \to 0,$$

Then

$$\varrho_h \to \varrho \textit{ (strongly) in } L^q\left((0,T)\times\Omega\right),$$

$$\boldsymbol{m}_h \to \boldsymbol{m} \textit{ (strongly) in } L^q\left((0,T)\times\Omega\right),\ 1 \leq q < \infty$$

where $(\varrho, \boldsymbol{m})$ *is the strong sol.*

Error estimates

- suitable tool: relative energy

$$\mathcal{E}(\varrho_h, \boldsymbol{m}_h | \varrho, \boldsymbol{u}) = \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_h |\boldsymbol{u}_h - \boldsymbol{u}|^2 + \mathbb{E}(\varrho_h|\varrho) \right) \, \mathrm{d}x,$$

$$\mathbb{E}(\varrho_h|\varrho) = \mathcal{P}(\varrho_h) - \mathcal{P}'(\varrho)(\varrho_h - \varrho) - \mathcal{P}(\varrho), \quad \mathcal{P}'(r) = p(r)$$

Error estimates

- suitable tool: relative energy

$$\mathcal{E}(\varrho_h, \boldsymbol{m}_h | \varrho, \boldsymbol{u}) = \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_h |\boldsymbol{u}_h - \boldsymbol{u}|^2 + \mathbb{E}(\varrho_h | \varrho) \right) \, \mathrm{d}x,$$
$$\mathbb{E}(\varrho_h|\varrho) = \mathcal{P}(\varrho_h) - \mathcal{P}'(\varrho)(\varrho_h - \varrho) - \mathcal{P}(\varrho), \quad \mathcal{P}'(r) = p(r)$$

Theorem (Feireisl, ML, She (Num.Math. '22))

Let $\varrho_0 \in W^{6,2}(\Omega),\ 0 < \underline{\varrho} \leq \min \varrho_0,\ \boldsymbol{m}_0 \in W^{6,2}(\Omega)$ *and*

$$\|\varrho_h\|_{L^\infty((0,T)\times\Omega)} \leq \overline{\varrho} \quad \textit{and} \quad \|\boldsymbol{u}_h\|_{L^\infty((0,T)\times\Omega)} \leq \overline{u}.$$

Then

$$\sup_{0 \leq t \leq \tau} \int_\Omega \mathcal{E}(\varrho_h, \boldsymbol{m}_h | \varrho, \boldsymbol{u}) \mathrm{d}x + \mu \int_0^T \int_\Omega |\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 \mathrm{d}x \, \mathrm{d}t \leq C(h + \Delta t)$$

Error estimates

- suitable tool: relative energy

$$\mathcal{E}(\varrho_h, \boldsymbol{m}_h | \varrho, \boldsymbol{u}) = \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_h |\boldsymbol{u}_h - \boldsymbol{u}|^2 + \mathbb{E}(\varrho_h|\varrho) \right) \, \mathrm{d}x,$$
$$\mathbb{E}(\varrho_h|\varrho) = \mathcal{P}(\varrho_h) - \mathcal{P}'(\varrho)(\varrho_h - \varrho) - \mathcal{P}(\varrho), \quad \mathcal{P}'(r) = p(r)$$

Theorem (Feireisl, ML, She (Num.Math. '22))

Let $\varrho_0 \in W^{6,2}(\Omega),\ 0 < \underline{\varrho} \leq \min \varrho_0,\ \boldsymbol{m}_0 \in W^{6,2}(\Omega)$ *and*

$$\|\varrho_h\|_{L^\infty((0,T)\times\Omega)} \leq \overline{\varrho} \quad \textit{and} \quad \|\boldsymbol{u}_h\|_{L^\infty((0,T)\times\Omega)} \leq \overline{u}.$$

Then

$$\sup_{0 \leq t \leq \tau} \int_\Omega \mathcal{E}(\varrho_h, \boldsymbol{m}_h | \varrho, \boldsymbol{u}) \mathrm{d}x + \mu \int_0^T \int_\Omega |\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 \mathrm{d}x \, \mathrm{d}t \leq C(h + \Delta t)$$
$$\|\varrho_h(\tau) - \varrho(\tau)\|_{L^2(\Omega)} + \|\boldsymbol{m}_h(\tau) - \boldsymbol{m}(\tau)\|_{L^2(\Omega)} \leq C(\sqrt{h} + \sqrt{\Delta t}),\ \tau \in (0,T].$$

where $C = C(T, \|(\varrho_0, \boldsymbol{u}_0)\|_{W^{6,2}}, \underline{\varrho}, \overline{\varrho}, \overline{u})$.

Statistical sols.

- probability space $\left[\Gamma, \mathcal{B}, \mathcal{P}\right]$
- data are random variables in D

$$D = \left\{ [\varrho_0, \boldsymbol{m}_0, \mu, \lambda] \middle| \ \varrho_0 \in W^{3,2}(\Omega), \ \boldsymbol{m}_0 \in W^{3,2}(\Omega), \right.$$
$$\left. \min_{\Omega} \varrho_0 > 0, \mu \geq \underline{\mu} > 0, \ \lambda \geq 0 \right\}$$

$\boldsymbol{U}_0 : \omega \in \Gamma \mapsto [\varrho_0(\omega), \boldsymbol{m}_0(\omega), \mu(\omega), \lambda(\omega)] \in D$ is Borel measurable

Statistical sols.

- probability space $\Big[\Gamma, \mathcal{B}, \mathcal{P}\Big]$
- data are random variables in D

$$D = \Big\{ [\varrho_0, \boldsymbol{m}_0, \mu, \lambda] \Big| \ \varrho_0 \in W^{3,2}(\Omega), \ \boldsymbol{m}_0 \in W^{3,2}(\Omega), \min_{\Omega} \varrho_0 > 0, \mu \geq \underline{\mu} > 0, \ \lambda \geq 0 \Big\}$$

$\boldsymbol{U}_0 : \omega \in \Gamma \mapsto [\varrho_0(\omega), \boldsymbol{m}_0(\omega), \mu(\omega), \lambda(\omega)] \in D$ is Borel measurable

We set

$$\boldsymbol{U}(t, \omega) := \boldsymbol{U}\left[\boldsymbol{U}_0(\omega); t\right], \ t \in [0, T_{\max}(\omega)), \quad a.a.\ \omega \in \Gamma$$

$\boldsymbol{U} = (\varrho, \boldsymbol{m})$ is the strong statistical solution

Following J. Nash...

Principal Hypothesis: Numerical solutions are bounded in probability.

for any $\varepsilon > 0,$ there exists $M = M(\varepsilon)$ such that for all $h \in (0,1)$

$$\mathcal{P}\left(\left[\|\varrho_h\|_{L^\infty((0,T)\times\Omega)} + \|\boldsymbol{u}_h\|_{L^\infty((0,T)\times\Omega)} > M\right]\right) \leq \varepsilon.$$

Statistical sol. / Convergence of Monte Carlo FVM

Theorem (Feireisl, ML, She, Yuan (M³AS '22))

Let $\left[\varrho_0, \boldsymbol{m}_0, \mu, \lambda\right] \in D$ *be random data*

$$\varrho_0 \in W^{3,2}(\Omega),\ \min \varrho_0 = \underline{\varrho} > 0,\ \boldsymbol{m}_0 \in W^{3,2}(\Omega) \quad \mathcal{P} - \text{a.s.}$$

Let $[\varrho_0^n, \boldsymbol{m}_0^n, \mu^n, \lambda^n]$, $n = 1, 2, \ldots, N$ *be i.i.d. copies and* $(\varrho_h^n, \boldsymbol{m}_h^n)_{h \searrow 0}$ *FV solutions, that are bounded in probability.*

Statistical sol. / Convergence of Monte Carlo FVM

Theorem (Feireisl, ML, She, Yuan (M^{3}AS '22))

Let $\left[\varrho_0, \boldsymbol{m}_0, \mu, \lambda\right] \in D$ *be random data*

$$\varrho_0 \in W^{3,2}(\Omega),\ \min \varrho_0 = \underline{\varrho} > 0,\ \boldsymbol{m}_0 \in W^{3,2}(\Omega) \quad \mathcal{P} - a.s.$$

Let $[\varrho_0^n, \boldsymbol{m}_0^n, \mu^n, \lambda^n]$, $n = 1, 2, \dots, N$ *be i.i.d. copies and* $(\varrho_h^n, \boldsymbol{m}_h^n)_{h \searrow 0}$ *FV solutions, that are bounded in probability.*

Then for $N \to \infty,\ h \to 0$

$$\mathbb{E}\left[\left\| \frac{1}{N}\sum_{n=1}^{N}\left(\varrho_h^n - \mathbb{E}\left[\varrho\right]\right)\right\|^q_{L^r(0,T;L^\gamma(\Omega))}\right] \to 0 \quad \text{for} \quad 1 \le q < 2\gamma,$$

$$\mathbb{E}\left[\left\| \frac{1}{N}\sum_{n=1}^{N}\left(\boldsymbol{m}_h^n - \mathbb{E}\left[\boldsymbol{m}\right]\right)\right\|^q_{L^r(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))}\right] \to 0 \quad \text{for} \ 1 \le q < \frac{4\gamma}{\gamma+1},$$

where $(\varrho, \boldsymbol{m} = \varrho \boldsymbol{u})$ *is the strong statistical solution and* $1 \le r < \infty$.

Main ideas of the proof

- FV solutions $(\varrho_{h_k}, \boldsymbol{u}_{h_k})_{h_k \searrow 0}$ ... a family of random variables

$$\Big[\varrho_0, \boldsymbol{m}_0, \mu, \lambda, \varrho_{h_k}, \boldsymbol{u}_{h_k}, \Lambda_{h_k}\Big]_{h_k \searrow 0} \qquad \Lambda_{h_k} = \|(\varrho_{h_k}, \boldsymbol{u}_{h_k})\|_{L^\infty_{t,x}}$$

ranging in the Polish space

$$Y = W^{3,2}(\Omega) \times W^{3,2}(\Omega) \times R \times R \times W^{-k,2}_{t,x} \times W^{-k,2}_{t,x} \times R, \ k \geq 4$$

Main ideas of the proof

- FV solutions $(\varrho_{h_k}, \boldsymbol{u}_{h_k})_{h_k \searrow 0}$ ... a family of random variables

$$\Big[\varrho_0, \boldsymbol{m}_0, \mu, \lambda, \varrho_{h_k}, \boldsymbol{u}_{h_k}, \Lambda_{h_k}\Big]_{h_k \searrow 0} \qquad \Lambda_{h_k} = \|(\varrho_{h_k}, \boldsymbol{u}_{h_k})\|_{L^\infty_{t,x}}$$

  ranging in the Polish space

$$Y = W^{3,2}(\Omega) \times W^{3,2}(\Omega) \times R \times R \times W^{-k,2}_{t,x} \times W^{-k,2}_{t,x} \times R, \ k \geq 4$$

- the family of laws associated to above family is tight in Y

Main ideas of the proof

- FV solutions $(\varrho_{h_k}, \boldsymbol{u}_{h_k})_{h_k \searrow 0}$ ... a family of random variables

$$\Big[\varrho_0, \boldsymbol{m}_0, \mu, \lambda, \varrho_{h_k}, \boldsymbol{u}_{h_k}, \Lambda_{h_k}\Big]_{h_k \searrow 0} \qquad \Lambda_{h_k} = \|(\varrho_{h_k}, \boldsymbol{u}_{h_k})\|_{L^\infty_{t,x}}$$

  ranging in the Polish space

$$Y = W^{3,2}(\Omega) \times W^{3,2}(\Omega) \times R \times R \times W^{-k,2}_{t,x} \times W^{-k,2}_{t,x} \times R, \ k \geq 4$$

- the family of laws associated to above family is tight in Y
- Skorokhod representation thm. ... there is a new probability space/family of random var. equivalent in law

Main ideas of the proof

- FV solutions $(\varrho_{h_k}, \boldsymbol{u}_{h_k})_{h_k \searrow 0}$ ... a family of random variables

$$\Big[\varrho_0, \boldsymbol{m}_0, \mu, \lambda, \varrho_{h_k}, \boldsymbol{u}_{h_k}, \Lambda_{h_k}\Big]_{h_k \searrow 0} \qquad \Lambda_{h_k} = \|(\varrho_{h_k}, \boldsymbol{u}_{h_k})\|_{L^\infty_{t,x}}$$

ranging in the Polish space

$$Y = W^{3,2}(\Omega) \times W^{3,2}(\Omega) \times R \times R \times W^{-k,2}_{t,x} \times W^{-k,2}_{t,x} \times R, \; k \geq 4$$

- the family of laws associated to above family is tight in Y
- Skorokhod representation thm. ... there is a new probability space/family of random var. equivalent in law

$$\Big[\tilde{\varrho}_{0,h_k}, \widetilde{\boldsymbol{m}}_{0,h_k}, \widetilde{\mu}_{h_k}, \widetilde{\lambda}_{h_k}, \widetilde{\varrho}_{h_k}, \widetilde{\boldsymbol{u}}_{h_k}, \widetilde{\Lambda}_{h_k}\Big] \sim \Big[\varrho_0, \boldsymbol{m}_0, \mu, \lambda, \varrho_{h_k}, \boldsymbol{u}_{h_k}, \Lambda_{h_k}\Big]$$

$$\widetilde{\Lambda}_{h_k} = \|(\widetilde{\varrho}_{h_k}, \widetilde{\boldsymbol{u}}_{h_k})\|_{L^\infty_{t,x}} < \infty,$$

$$\tilde{\varrho}_{0,h_k} \to \tilde{\varrho}_0, \quad \widetilde{\boldsymbol{m}}_{0,h_k} \to \widetilde{\boldsymbol{m}}_0 \text{ in } W^{3,2}(\Omega),$$

$$\widetilde{\mu}_{h_k} \to \widetilde{\mu}, \; \widetilde{\lambda}_h \to \widetilde{\lambda},$$

$$\tilde{\varrho}_{h_k} \to \tilde{\varrho}, \qquad \widetilde{\boldsymbol{u}}_{h_k} \to \widetilde{\boldsymbol{u}} \text{ in } L^r_{t,x}, \; 1 \leq r < \infty \qquad \text{a.s}$$

where $(\tilde{\varrho}, \widetilde{\boldsymbol{u}})$ is the strong statistical sol.

- strong sol. (convergence of the whole seq.) ... Gyöngy–Krylov thm.

$$\|\varrho_h - \varrho\|_{L^r_{t,x}} \to 0 \qquad \|\boldsymbol{m}_h - \boldsymbol{m}\|_{L^r_{t,x}} \to 0 \text{ in probability}, \ 1 \leq r < \infty$$

Statistical sol./ Error estimates for Monte Carlo FVM

Theorem (Feireisl, ML, She, Yuan (M^3AS'22))

Let $\left[\varrho_0, \boldsymbol{m}_0, \mu, \lambda\right] \in D$ *be random data*

$$(\varrho_0, \boldsymbol{u}_0) \in W^{6,2}(\Omega) \times W^{6,2}(\Omega),$$

$\min \varrho_0 \equiv \underline{\varrho} > 0, \quad \mathcal{P} -$ *a.s.*

Suppose that $[\varrho_0^n, \boldsymbol{m}_0^n, \mu^n, \lambda^n]$, $n = 1, 2, \dots$ *are i.i.d. copies of random data.*
Let $(\varrho_h^n, \boldsymbol{m}_h^n)_{h \searrow 0}$ *be FV solutions that are bounded in probability.*

Statistical sol./ Error estimates for Monte Carlo FVM

Theorem (Feireisl, ML, She, Yuan (M³AS'22))

Let $\left[\varrho_0, \boldsymbol{m}_0, \mu, \lambda\right] \in D$ *be random data*

$$(\varrho_0, \boldsymbol{u}_0) \in W^{6,2}(\Omega) \times W^{6,2}(\Omega),$$

$\min \varrho_0 \equiv \underline{\varrho} > 0, \quad \mathcal{P} - a.s.$

Suppose that $[\varrho_0^n, \boldsymbol{m}_0^n, \mu^n, \lambda^n]$, $n = 1, 2, \dots$ *are i.i.d. copies of random data.*
Let $(\varrho_h^n, \boldsymbol{m}_h^n)_{h \searrow 0}$ *be FV solutions that are bounded in probability.*

Then for the expectation of the statistical errors

$$\mathbb{E}\left[\left\| \frac{1}{N} \sum_{n=1}^{N} \Big(\varrho^n(t,\cdot) - \mathbb{E}\left[\varrho(t,\cdot)\right] \Big) \right\|_{L^\gamma(\Omega)}^{r} \right] \lesssim N^{1-r} \quad \text{for} \quad r = \min(2, \gamma)$$

$$\mathbb{E}\left[\left\| \frac{1}{N} \sum_{n=1}^{N} \Big(\boldsymbol{m}^n(t,\cdot) - \mathbb{E}\left[\boldsymbol{m}(t,\cdot)\right] \Big) \right\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)}^{\frac{2\gamma}{\gamma+1}} \right] \lesssim N^{\frac{1-\gamma}{1+\gamma}}, \ t \in [0,T].$$

Statistical sol./ Error estimates for Monte Carlo FVM

Theorem (cont.)

The approximation errors are estimated in probability: for any $\varepsilon > 0$, there exists $K(\varepsilon)$

$$\mathcal{P}\left[\left(\left\| \frac{1}{N} \sum_{n=1}^{N} (\varrho_h^n(t,\cdot) - \varrho^n(t,\cdot)) \right\|_{L^2(\Omega)} + \left\| \frac{1}{N} \sum_{n=1}^{N} (\boldsymbol{m}_h^n(t,\cdot) - \boldsymbol{m}^n(t,\cdot)) \right\|_{L^2(\Omega)} \right) \le K(\varepsilon)(\sqrt{h} + \sqrt{\Delta t}) \right] \ge 1 - \varepsilon \qquad \text{for all } t \in [0,T],$$

$$h \in (0,1), \quad \Delta t \in (0,1), \quad N = 1, 2, \ldots .$$

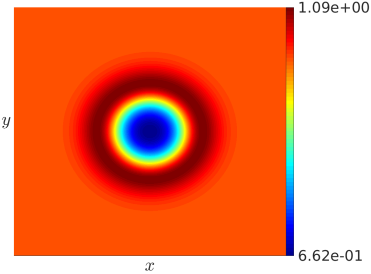
Random perturbation of the vortex interface

$$\varrho_0(x) = 1$$

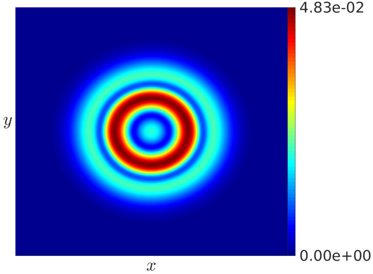
$$\boldsymbol{u}_0(x) = \begin{cases} \left(\dfrac{[1-\cos(2\pi|x|/I)]x_2}{|x|},\ -\dfrac{[1-\cos(2\pi|x|/I)]/I)x_1}{|x|} \right) & |x| < I, \\ (0,0) & \text{otherwise} \end{cases}$$

where

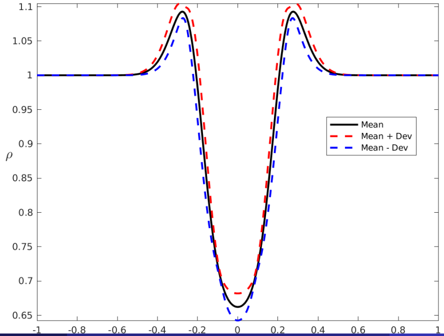
$$I = 0.5 + Y(\omega), \quad Y \ \sim \ \mathcal{U}\left(-0.1, 0.1\right), \quad j = 1, 2, 3.$$



(a) ϱ - **Mean**



(b) ϱ - **Deviation**



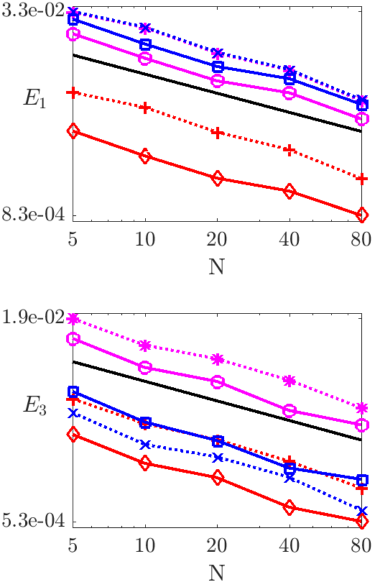


Figure: Statistical convergence for mean and devation

Navier-Stokes-Fourier system

Theorem (Feireisl, ML, Mizerová, She (IMA J. Num. Anal.'20), Basaric, ML, Mizerová, She, Yuan (MathComp'23))

Let $(\varrho_0, \vartheta_0, \boldsymbol{u}_0) \in W^{3,2}(\Omega)$*. Let* $(\varrho_h, \vartheta_h, \boldsymbol{u}_h)_{h \searrow 0}$ *be the FVM solution, s.t. for all* $h \to 0$ *and* $t \in (0,T)$

$$0 < \underline{\varrho} \le \varrho_h \le \overline{\varrho} \qquad 0 < \underline{\vartheta} \le \vartheta_h \le \overline{\vartheta}$$

Then

$$(\varrho_h, \vartheta_h, \boldsymbol{u}_h) \to (\varrho, \vartheta, \boldsymbol{u}) \text{ (strongly) in } L^p\left((0,T) \times \Omega\right),\ p \in [1, \infty),$$

Navier-Stokes-Fourier system

Theorem (Feireisl, ML, Mizerová, She (IMA J. Num. Anal.'20), Basaric, ML, Mizerová, She, Yuan (MathComp'23))

Let $(\varrho_0, \vartheta_0, \boldsymbol{u}_0) \in W^{3,2}(\Omega)$. *Let* $(\varrho_h, \vartheta_h, \boldsymbol{u}_h)_{h \searrow 0}$ *be the FVM solution, s.t. for all* $h \to 0$ *and* $t \in (0,T)$

$$0 < \underline{\varrho} \le \varrho_h \le \overline{\varrho} \qquad 0 < \underline{\vartheta} \le \vartheta_h \le \overline{\vartheta}$$

Then

$$(\varrho_h, \vartheta_h, \boldsymbol{u}_h) \to (\varrho, \vartheta, \boldsymbol{u}) \text{ (strongly) in } L^p\left((0,T) \times \Omega\right),\ p \in [1, \infty),$$

If $(\varrho_0, \vartheta_0, \boldsymbol{u}_0) \in W^{6,2}(\Omega)$ *then*

$$\begin{aligned}
&\|\varrho_h - \varrho\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\vartheta_h - \vartheta\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))} + \|\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \vartheta\|_{L^2(0,T;L^2(\Omega))} \\
&\le c(\Delta t^{1/2} + h^{1/4}),
\end{aligned}$$

where ϱ, ϑ, *and* $\boldsymbol{u}$ *is the strong solution of the Navier–Stokes–Fourier system.*

Error analysis of MC FVM for NSF system

Theorem (ML, She, Yuan (arXiv'24))

Let the data be random variables $\boldsymbol{U}_0 = \left[\varrho_0, \boldsymbol{u}_0, \vartheta_0, \mu, \lambda, \kappa\right] \in D \quad \mathcal{P}$*-a.s.,*

$(\varrho_0, \boldsymbol{u}_0, \vartheta_0) \in W^{6,2}(\Omega)$ *and bdd. total energy and entropy in expectation.*
Let $\mathbf{U}_0^n$, $n = 1, 2, \dots$ *are i.i.d. copies of random data and* $(\varrho_h^n, \boldsymbol{u}_h^n, \vartheta_h^n)_{h \searrow 0}$ *be a sequence of corresponding FV solutions, bounded in probability.*
Then MC estimators satisfy for all $t \in [0, T]$ *and* $N = 1, 2, \dots$

$$\left\| \frac{1}{N} \sum_{n=1}^{N} (\varrho^n, \boldsymbol{m}^n, S^n)(t, \cdot) - \mathbb{E}\left[(\varrho, \boldsymbol{m}, S)(t, \cdot)\right] \right\|_{W^{-k,2}(\Omega)} \lesssim N^{-1/2}, \quad k > \frac{d}{2}$$

$\mathcal{P}$ *– a.s., and for any* $\varepsilon > 0$, *there exists* $K = K(\varepsilon, N)$, *such that*

$$\mathcal{P}\left[\left\| \frac{1}{N} \sum_{n=1}^{N} \left(\varrho^n, \boldsymbol{m}^n, S^n\right)(t, \cdot) - \frac{1}{N} \sum_{n=1}^{N} \left(\varrho_h^n, \boldsymbol{m}_h^n, S_h^n\right)(t, \cdot) \right\|_{L^2(\Omega)} \leq K \left(\Delta t^{1/2} + h^{1/4} \right) \right] \geq 1 - \varepsilon$$

Raleigh-Bénard problem

- initial data: [Feireisl, ML, She, Yuan: FoCoM ('24)]

$$\varrho_0(x) = 1.2 + (1 + Y_1(\omega)) \sin\left(\frac{\pi x_2}{2}\right), \ (Y_1, Y_2) \sim \mathcal{U}([-0.1, 0.1]^2)$$

$$\vartheta_0(x) = a + b\, x_2 + cP(x_1)\sin(\pi x_2) + Y_2(\omega)\sin(\pi x_1)\sin\left(\frac{\pi(x_2+1)}{4}\right)$$

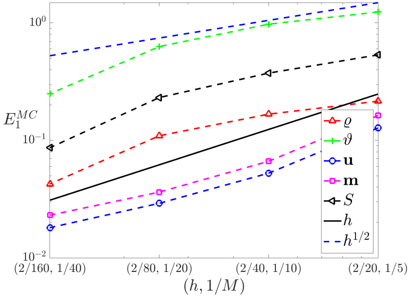


Figure: **MCFV Total errors** of the expectation $E^{MC}(U, h, M(h))$ with $(h, 1/M(h)) = (h, \mathcal{O}(h)) = (2/(20 \cdot 2^l),\ 1/(5 \cdot 2^l)),\ l = 0, \dots, 3$. The blue dashed and black solid lines denote the reference lines of $h^{1/2}$ and h

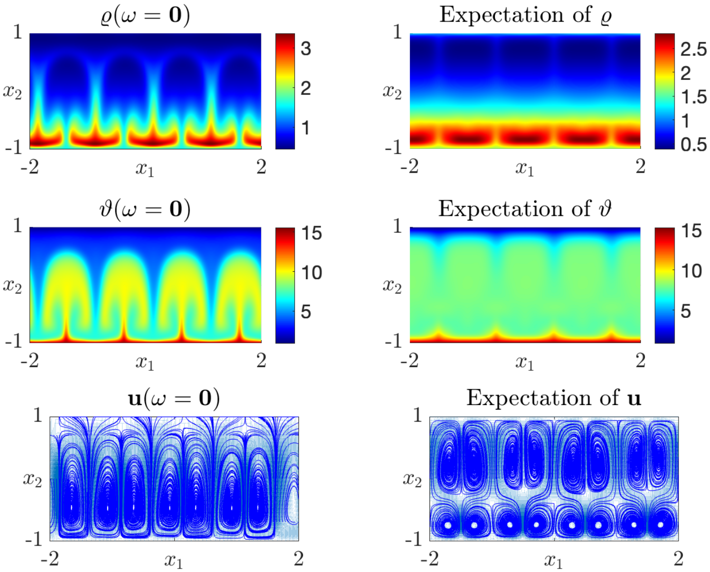


Figure: Rayleigh–Bénard experiment: the deterministic FV solution (left) and the **MCFV solutions** (right) at time $T = 8$ on a mesh with 640×320 cells.

Euler system

Good and bad news:

- local-in-time well-posedness in class $C([0,T]; W^{k,2}(\Omega))$, $k > 1 + \frac{d}{2}$
  Majda ('84)
- non-unique weak admissible solutions: (energy dissipative/entropy stable) De Lellis, Székelyhidi ('10), Buckmaster, DeLellis, Székelyhidi ('19), Buckmaster, Vicol ('19), Chiordarolli, Feireisl ('24)

Euler system

Good and bad news:

- local-in-time well-posedness in class $C([0,T]; W^{k,2}(\Omega))$, $k > 1 + \frac{d}{2}$ Majda ('84)
- non-unique weak admissible solutions: (energy dissipative/entropy stable) De Lellis, Székelyhidi ('10), Buckmaster, DeLellis, Székelyhidi ('19), Buckmaster, Vicol ('19), Chiordarolli, Feireisl ('24)
- semigroup selection of global-in-time dissipative solutions that maximize energy dissipation /entropy production Breit, Feireisl, Hofmanová ('20)

$\mathcal{K}$-convergence: averaging over mesh resolutions

- Kómlós ('67), Banach-Sachs ('30)

Theorem (Feireisl, ML, Mizerová, She (Springer'21))

Let $[\varrho_0, \boldsymbol{m}_0] \in L^1(\Omega) \times L^1(\Omega)$, $\int_\Omega E(\varrho_0, \boldsymbol{m}_0) \lesssim 1$, $\varrho_0 > 0$.
Let $\{\varrho_h, \boldsymbol{m}_h\}_{h \searrow 0}$ *be a sequence of consistent approximation (viscosity FVM) of barotropic Euler equations.*

Then there is a $\{h_{n_k}\}_{k=1}^{\infty}$ *such that*

$$\frac{1}{K} \sum_{k=1}^{K} (\varrho_{h_{n_k}}, \boldsymbol{m}_{h_{n_k}}) \to (\varrho, \boldsymbol{m}) \text{ in } L^q((0,T) \times \Omega) \text{ as } K \to \infty$$

for any $1 < q \leq \frac{2\gamma}{\gamma + 1}$. *Here* $(\varrho, \boldsymbol{m}) \in \mathcal{U}(\varrho_0, \boldsymbol{m}_0)$ *is a dissipative sol. emanating from* $[\varrho_0, \boldsymbol{m}_0]$.

$\mathcal{K}$-convergence of Monte Carlo FVM

Theorem (Feireisl, ML, Mizerová, Yu (arXiv'24))

Let $(\varrho_0^n, \boldsymbol{m}_0^n)_{n=1}^{\infty}$ *be i.i.d. copies of random data* $[\varrho_0, \boldsymbol{m}_0] \in D$ *satisfying*

$$\frac{1}{L} \leq \varrho_0(x) \leq L, \ |\boldsymbol{m}_0(x)| \leq L \text{ for a.a. } x \in \Omega.$$

Let $\{\varrho_h^n, \boldsymbol{m}_h^n\}_{h \searrow 0}$ *be consistent approximations (viscosity FVM) of the barotropic Euler system.*

$\mathcal{K}$-convergence of Monte Carlo FVM

Theorem (Feireisl, ML, Mizerová, Yu (arXiv'24))

Let $(\varrho_0^n, \boldsymbol{m}_0^n)_{n=1}^{\infty}$ *be i.i.d. copies of random data* $[\varrho_0, \boldsymbol{m}_0] \in D$ *satisfying*

$$\frac{1}{L} \leq \varrho_0(x) \leq L, \ |\boldsymbol{m}_0(x)| \leq L \text{ for a.a. } x \in \Omega.$$

Let $\{\varrho_h^n, \boldsymbol{m}_h^n\}_{h\searrow 0}$ *be consistent approximations (viscosity FVM) of the barotropic Euler system.*
Then there is a subsequence $\{h_{m_k}\}_{k=1}^{\infty}$ *such that*

$$\mathbb{E}\left[\left\| \frac{1}{NK}\sum_{n=1}^{N}\sum_{k=1}^{K}(\varrho_{h_{m_k}}^n, \boldsymbol{m}_{h_{m_k}}^n) - \mathbb{E}\left[(\varrho, \boldsymbol{m})\right]\right\|_{L^q((0,T)\times\Omega)}\right] \to 0 \text{ as } K,\ N \to \infty$$

for any $1 < q \leq \dfrac{2\gamma}{\gamma+1}$*, where* $(\varrho, \boldsymbol{m}) \in \mathcal{U}(\varrho_0, \boldsymbol{m}_0)$ *is a measurable selection.*

Numerical convergance rates

- Césaro averages

$$\widetilde{\boldsymbol{U}}_{h_K} := \frac{1}{K} \sum_{k=1}^{K} \boldsymbol{U}_{h_{n_k}}$$

- errors

$$E_1 = \frac{1}{L} \sum_{\ell=1}^{L} \left\| \frac{1}{N} \sum_{n=1}^{N} \left(\boldsymbol{U}_h^{n,\ell}(T,\cdot) - \mathbb{E}[\boldsymbol{U}_h(T,\cdot)] \right) \right\|_{L^1(\Omega)},$$

$$E_2 = \frac{1}{L} \sum_{\ell=1}^{L} \left\| \frac{1}{N} \sum_{n=1}^{N} \left(\widetilde{\boldsymbol{U}}_{h_K}^{n,\ell}(T,\cdot) - \mathbb{E}[\widetilde{\boldsymbol{U}}_{h_K}(T,\cdot)] \right) \right\|_{L^1(\Omega)}$$

with $L = 20$, $N = N(h)$

$$\mathbb{E}[\boldsymbol{U}_h] = \frac{1}{N_{ref}} \sum_{s=1}^{N_{ref}} \boldsymbol{U}_h^s, \quad \mathbb{E}[\widetilde{\boldsymbol{U}}_{h_K}] = \frac{1}{N_{ref}} \sum_{s=1}^{N_{ref}} \widetilde{\boldsymbol{U}}_{h_K}^s, \quad N_{ref} = 100$$

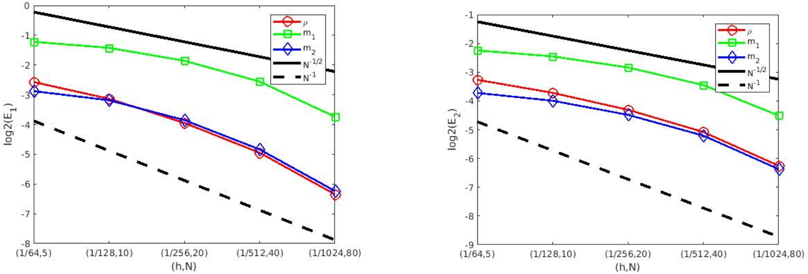


Figure: The errors E_1 and E_2 in L^1-norm with parmaters $h = 1/(2^{\ell+5})$, $N(h) = 5 \cdot 2^{\ell-1}, \ell = 1, 2, \ldots, 5$.

Concluding remarks

- dissipative solutions $\Longrightarrow$ elegant way to study convergence of nonlinear problems (even essentially ill-posed Euler eqs.)

Concluding remarks

- dissipative solutions $\Longrightarrow$ elegant way to study convergence of nonlinear problems (even essentially ill-posed Euler eqs.)
- **Lax-equivalence principle** for nonlinear problems:
  consistency & stability $\Longleftrightarrow$ convergence

Concluding remarks

- dissipative solutions $\Longrightarrow$ elegant way to study convergence of nonlinear problems (even essentially ill-posed Euler eqs.)
- **Lax-equivalence principle** for nonlinear problems:
  consistency & stability $\Longleftrightarrow$ convergence
- if the FV solution is uniformly bdd. w.r.t $h \searrow 0 \Longrightarrow$
  the strong solution exists

Concluding remarks

- dissipative solutions $\Longrightarrow$ elegant way to study convergence of nonlinear problems (even essentially ill-posed Euler eqs.)
- **Lax-equivalence principle** for nonlinear problems:
  consistency & stability $\Longleftrightarrow$ convergence
- if the FV solution is uniformly bdd. w.r.t $h \searrow 0 \Longrightarrow$
  the strong solution exists
- stochastic compactness techniques, Strong Law of large numbers, Central limit theorem $\Longrightarrow$ error analysis for random problems

DFG

TRR 146

SINO-GERMAN

G|FK

m3odel

Mainz Institute of Multiscale Modeling