On different constitutive relations and boundary conditions for fluids

Erika Maringová Kokavcová (ISTA)

Miroslav Bulíček, Josef Málek (Charles University)

Pablo Alexei Gazca Orozco (University of Freiburg),
Franz Gmeineder (University of Konstanz), Tabea Tscherpel (TU Darmstadt)

Institute of Science and Technology Austria

MPDE 2024
Prague, 24 September 2024

Outline

- implicitly constituted incompressible fluid problem
- typical problems, motivation
- our result on existence and uniqueness
- numerical analysis - ongoing work

Outline

- implicitly constituted incompressible fluid problem
- typical problems, motivation
- our result on existence and uniqueness
- numerical analysis - ongoing work

Bulíček, Málek, M: *On nonlinear parabolic problems with implicit constitutive equations involving flux*, M3AS (2021).

Bulíček, Málek, M: *On unsteady internal flows of incompressible fluids characterized by implicit constitutive equations in the bulk and on the boundary*, J. Math. Fluid Mech. (2023).

Gazca Orozco, Gmeineder, MK, Tscherpel, *in preparation 2024*.

Navier–Stokes-like problem

- flow of homogeneous, incompressible fluids

$$
\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} + \nabla p &= \boldsymbol{f} && \text{in } Q := (0,T) \times \Omega, \\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma := (0,T) \times \partial\Omega, \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

- Navier–Stokes: $\boldsymbol{S} = 2\nu \boldsymbol{D}\boldsymbol{v}$ in Q $\longrightarrow$ implicit: $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0$ in Q
- boundary condition: $\boldsymbol{s} := -(\boldsymbol{S}\boldsymbol{n})_\tau = \gamma \boldsymbol{v}_\tau$ on Γ $\longrightarrow$ implicit: $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_\tau) = 0$ on Γ

Navier–Stokes-like problem

- flow of homogeneous, incompressible fluids

$$
\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} + \nabla p &= \boldsymbol{f} && \text{in } Q := (0,T) \times \Omega, \\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma := (0,T) \times \partial\Omega, \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

- Navier–Stokes: $\boldsymbol{S} = 2\nu \boldsymbol{D}\boldsymbol{v}$ in Q $\longrightarrow$ implicit: $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0$ in Q
- boundary condition: $\boldsymbol{s} := -(\boldsymbol{S}\boldsymbol{n})_\tau = \gamma \boldsymbol{v}_\tau$ on Γ $\longrightarrow$ implicit: $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_\tau) = 0$ on Γ

  for example

$$
\boldsymbol{S} - |\boldsymbol{D}\boldsymbol{v}|^{r-2} \boldsymbol{D}\boldsymbol{v} = 0 \quad \text{or} \quad \boldsymbol{D}\boldsymbol{v} - (|\boldsymbol{S}| - \sigma_*)^+ \frac{\boldsymbol{S}}{|\boldsymbol{S}|} = 0
$$

Navier–Stokes-like problem

- flow of homogeneous, incompressible fluids

$$
\begin{aligned}
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} + \nabla p &= \boldsymbol{f} && \text{in } Q := (0,T) \times \Omega, \\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, \\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma := (0,T) \times \partial\Omega, \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

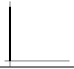
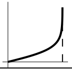
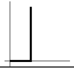
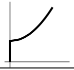
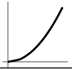
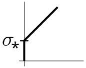
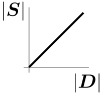
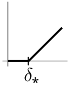
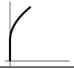
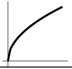
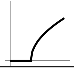
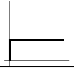
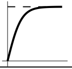
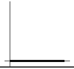
- Navier–Stokes: $\boldsymbol{S} = 2\nu \boldsymbol{D}\boldsymbol{v}$ in Q $\longrightarrow$ implicit: $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0$ in Q
- boundary condition: $\boldsymbol{s} := -(\boldsymbol{S}\boldsymbol{n})_\tau = \gamma \boldsymbol{v}_\tau$ on Γ $\longrightarrow$ implicit: $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_\tau) = 0$ on Γ
  for example

$$
\boldsymbol{S} - |\boldsymbol{D}\boldsymbol{v}|^{r-2}\boldsymbol{D}\boldsymbol{v} = 0 \quad \text{or} \quad \boldsymbol{D}\boldsymbol{v} - (|\boldsymbol{S}| - \sigma_*)^+ \frac{\boldsymbol{S}}{|\boldsymbol{S}|} = 0
$$

- weak formulation:

$$
\langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi} \rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{S} : \boldsymbol{D}\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{\varphi} \, \mathrm{d}S = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle,
$$

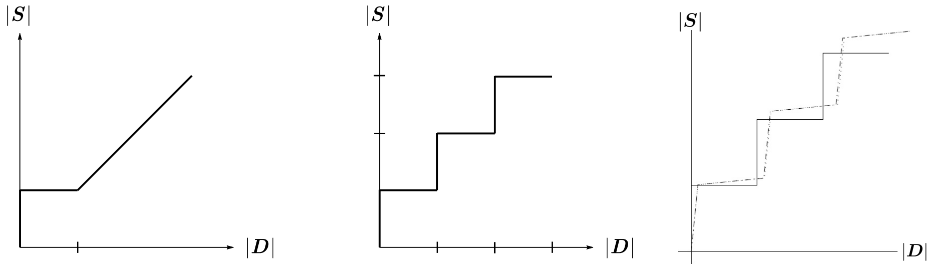
Power-law and/or activating effect

$r \to \infty$	rigid body 	limiting shear-rate 	Euler/limiting shear-rate 
$r \in (2, \infty)$	rigid/shear-thickening 	shear-thickening 	Euler/shear-thickening 
$r = 2$	Bingham = rigid/Navier-Stokes 	Navier-Stokes 	Euler/Navier-Stokes 
$r \in (1, 2)$	rigid/shear-thinning 	shear-thinning 	Euler/shear-thinning 
$r \to 1$	perfect plastic 	limiting shear stress 	Euler 
	$\|\boldsymbol{S}\| \le \sigma_* \iff \boldsymbol{Dv} = \boldsymbol{O}$	no activation	$\|\boldsymbol{Dv}\| \le \delta_* \iff \boldsymbol{S} = \boldsymbol{O}$

Blechta, Málek, Rajagopal ('20). Power-law $\boldsymbol{S} = |\boldsymbol{Dv}|^{r-2}\boldsymbol{Dv}$; activation $\boldsymbol{Dv} = (|\boldsymbol{S}| - \sigma_*)^+ \frac{\boldsymbol{S}}{|\boldsymbol{S}|}$.

Implicit constitutive theory

- Minty (1962): monotone mappings theory
- Rajagopal (2003, 2006): physical motivation and characterization of implicit theory
- Bulíček, Gwiazda, Málek, Świerczewska-Gwiazda (2009, 2012): mathematical analysis for graphs with selection: $(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) \in \mathcal{A}$
  proof: approximation by convolution
- Bulíček, Málek, M. (2021, 2023): $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0$, no selection needed, algebraical approximation, easy-to-verify assumptions



Maximal monotone r-graph $\mathcal{A} \subset \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$, $r \in (1,\infty)$, $r' := \frac{r}{r-1}$

(A1) $(0,0) \in \mathcal{A}$,

(A2) monotonicity: for any $(\boldsymbol{S}_1, \boldsymbol{D}_1), (\boldsymbol{S}_2, \boldsymbol{D}_2) \in \mathcal{A}$,

$$(\boldsymbol{S}_1 - \boldsymbol{S}_2) : (\boldsymbol{D}_1 - \boldsymbol{D}_2) \geq 0,$$

(A3) maximality: if for some $(\boldsymbol{S}, \boldsymbol{D})$ and all $(\overline{\boldsymbol{S}}, \overline{\boldsymbol{D}}) \in \mathcal{A}$

$$(\boldsymbol{S} - \overline{\boldsymbol{S}}) : (\boldsymbol{D} - \overline{\boldsymbol{D}}) \geq 0$$

holds, then $(\boldsymbol{S}, \boldsymbol{D}) \in \mathcal{A}$,

(A4) r-growth & coercivity: there exist $C_1, C_2 > 0$ such that for all $(\boldsymbol{S}, \boldsymbol{D}) \in \mathcal{A}$ there holds

$$\boldsymbol{S} : \boldsymbol{D} \geq C_1(|\boldsymbol{S}|^{r'} + |\boldsymbol{D}|^{r}) - C_2.$$

Implicit function $\mathcal{G} : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$

(G1) $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}) \in \mathcal{C}^{0,1}$ and $\mathcal{G}(0,0) = 0$,

(G2) for almost all $(\boldsymbol{S}, \boldsymbol{D})$:

$$\mathcal{G}_{\boldsymbol{S}} \geq 0, \quad \mathcal{G}_{\boldsymbol{D}} \leq 0, \quad \mathcal{G}_{\boldsymbol{D}}(\mathcal{G}_{\boldsymbol{S}})^T \leq 0, \quad \mathcal{G}_{\boldsymbol{S}} - \mathcal{G}_{\boldsymbol{D}} > 0,$$

(G3)

$$\begin{aligned} &\text{either} \quad \forall \boldsymbol{D} \quad \liminf_{|\boldsymbol{S}|\to+\infty} \mathcal{G}(\boldsymbol{S}, \boldsymbol{D}) : \boldsymbol{S} > 0 \\ &\text{or} \qquad \forall \boldsymbol{S} \quad \limsup_{|\boldsymbol{D}|\to+\infty} \mathcal{G}(\boldsymbol{S}, \boldsymbol{D}) : \boldsymbol{D} < 0, \end{aligned}$$

(G4) there exist $c_1, c_2 > 0$ such that for all $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}) = 0$ we have

$$\boldsymbol{S} : \boldsymbol{D} \geq c_1(|\boldsymbol{S}|^{r'} + |\boldsymbol{D}|^r) - c_2.$$

Existence of weak solution

Denote $z := \max\{r, q, \frac{(d+2)r}{(d+2)r-2d}\}$. For any

- $\Omega \in \mathcal{C}^{0,1}$, $T > 0$, $\boldsymbol{v}_0 \in H$, $\boldsymbol{f} \in L^{r'}(0, T; (W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega))^*)$
- $\mathcal{G}$ satisfying $(G1) - (G4)$ with $r > \frac{2d}{d+2}$,
- $\boldsymbol{g}$ satisfying $(g1) - (g4)$ with $q > 1$,

there exist $(\boldsymbol{v}, \boldsymbol{S}, \boldsymbol{s})$ a weak solution to the problem (1) with implicit relations

$$\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0 \qquad \text{in } Q,$$
$$\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_\tau) = 0 \qquad \text{on } \Gamma,$$

such that

$$\boldsymbol{v} \in L^r(0, T; W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap \mathcal{C}_w([0, T]; L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap L^q(\Gamma),$$
$$\partial_t \boldsymbol{v} \in L^{z'}(0, T; (W^{1,z}_{\boldsymbol{n},\mathrm{div}}(\Omega))^*),$$
$$\boldsymbol{S} \in L^{r'}(Q),\ \boldsymbol{s} \in L^{q'}(\Gamma).$$

Existence of weak solution

Denote $z := \max\{r, q, \frac{(d+2)r}{(d+2)r-2d}\}$. For any

- $\Omega \in \mathcal{C}^{0,1}$, $T > 0$, $\boldsymbol{v}_0 \in H$, $\boldsymbol{f} \in L^{r'}(0,T;(W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega))^*)$
- $\mathcal{G}$ satisfying $(G1)-(G4)$ with $r > \frac{2d}{d+2}$,
- $\boldsymbol{g}$ satisfying $(g1)-(g4)$ with $q > 1$,

there exist $(\boldsymbol{v}, \boldsymbol{S}, \boldsymbol{s})$ a weak solution to the problem (1) with implicit relations

$$\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v}) = 0 \qquad \text{in } Q,$$
$$\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_\tau) = 0 \qquad \text{on } \Gamma,$$

such that

$$\boldsymbol{v} \in L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap \mathcal{C}_w([0,T];L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap L^q(\Gamma),$$
$$\partial_t \boldsymbol{v} \in L^{z'}(0,T;(W^{1,z}_{\boldsymbol{n},\mathrm{div}}(\Omega))^*),$$
$$\boldsymbol{S} \in L^{r'}(Q),\ \boldsymbol{s} \in L^{q'}(\Gamma).$$

Also, if $\Omega \in \mathcal{C}^{1,1}$ and $\boldsymbol{f} \in L^{r'}(0,T;(W^{1,r}_{\boldsymbol{n}}(\Omega))^*)$, there exists a pressure $p \in L^{z'}(0,T;L^{z'}(\Omega))$.

On uniqueness

+ parabolic, no convective term: $r > 1$ (Bulíček, Málek, Maringová 2021)
+ power-law-fluid:

$$r \geq \frac{d+2}{2} \quad \text{(Ladyzhenskaya 1960s)}$$

$$r \geq \frac{11}{5}, \; d = 3 \;\; \& \;\; \text{add. growth cond.} \quad \text{(Bulíček, Kaplický, Pražák 2019)}$$

+ implicit fluids:

$$r \geq \frac{3d+2}{d+2} \quad \text{(Bulíček, Málek, Maringová 2023)}$$

On uniqueness

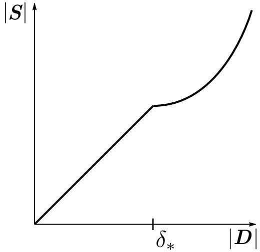
+ parabolic, no convective term: $r > 1$ (Bulíček, Málek, Maringová 2021)
+ power-law-fluid:

$$r \geq \frac{d+2}{2} \quad \text{(Ladyzhenskaya 1960s)}$$

$$r \geq \frac{11}{5},\ d = 3\ \ \&\ \ \text{add. growth cond.} \quad \text{(Bulíček, Kaplický, Pražák 2019)}$$

+ implicit fluids:

$$r \geq \frac{3d+2}{d+2} \quad \text{(Bulíček, Málek, Maringová 2023)}$$



Navier-Stokes prior to activation,
i.e. Navier-Stokes for $|\boldsymbol{Dv}| \leq \delta_*$, $\delta_* > 0$ arbitrary
and power-law for $|\boldsymbol{Dv}| > \delta_*$

On non-uniqueness

- for very weak solutions

$$r < \frac{3d+2}{d+2} \quad \text{(Burczak, Modena, Székelyhidi 2021)}$$

- for Leray solutions $r = 2$ (Albritton, Brué, Colombo 2022)

Numerical analysis - boundary conditions

BMM '21: parabolic problem, $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v})$, $r > 1$;

BMM '23: Navier–Stokes-like, $\mathcal{G}(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{v})$, $r > \frac{2d}{d+2}$; $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v})$, $q > 1$;

Numerical analysis - boundary conditions

BMM '21: parabolic problem, $\mathcal{G}(\boldsymbol{S}, \boldsymbol{Dv})$, $r > 1$;

BMM '23: Navier–Stokes-like, $\mathcal{G}(\boldsymbol{S}, \boldsymbol{Dv})$, $r > \frac{2d}{d+2}$; $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v})$, $q > 1$;

GO-G-MK-T $\sim$*'24*: Navier–Stokes, $\boldsymbol{S} = 2\nu \boldsymbol{Dv}$, $r = 2$; different BC, $\boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v})$, $1 \leq q < \cdot$

- existence of discrete and weak solution
- stationary / time dependent problem
- explicit / implicit BC
- coercive / non-coercive BC
- monotone / non-monotone BC

Numerical analysis - boundary conditions

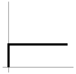
In particular:

- Nitsche penalisation due to approximation of the domain - cannot impose the BC directly
- Korn inequality with normal traces (under geometric assumption on the domain)

$$\|\nabla \boldsymbol{v}\|_{L^p(\Omega)} \lesssim \|\boldsymbol{D}\boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \quad \text{where } \Gamma \subset \partial\Omega$$

  (see Bauer, Pauly 2016 for $\boldsymbol{v}\cdot\boldsymbol{n} = 0$)

- Tresca slip: $q = 1$



- (explicit) non-monotone BC:

$$(\boldsymbol{s}^*(\boldsymbol{v}_1) - \boldsymbol{s}^*(\boldsymbol{v}_2)) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq -\lambda|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 \quad \text{for some } \lambda \geq 0$$

- dynamic slip (see Abbatiello, Bulíček, M. 2021):

$$\boldsymbol{s} = \alpha\boldsymbol{v} + \beta\partial_t\boldsymbol{v} \quad \text{for some } \alpha, \beta \geq 0$$