On the Solutions of Nonlinear Robin Boundary Value Problem

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Introduction and setting of the Problem

We consider a Robin problem driven by a nonlinear nonhmogeneous differential operator plus an indefinite potential term:

$$\begin{cases} -\text{div}\ a(Du(z)) + \xi(z)(u(z))^{p-1} = c(u(z))^{\tau-1} + \lambda f(z,u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)(u(z))^{p-1} = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ u > 0,\ 1 < \tau < p. \end{cases}$$

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In this problem, the map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the differential operator of (p_λ), is strictly monotone and continuous (thus, maximal monotone too) and satisfies certain other regularity and growth conditions listed in hypotheses $\widehat{H}$ below.
In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative corresponding to the map $a(.)$. If $u \in C^1(\bar{\Omega})$ then

$$\frac{\partial u}{\partial n_a} = \langle a(D(u)), n \rangle_{\mathbb{R}^N},$$

with $n(.)$ being the outward unit normal. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geq 0$.

$$-\text{div } a(Du(z)) + \xi(z)(u(z))^{p-1} = c(u(z))^{\tau-1} + \lambda f(z, u(z))$$

- The reaction term $f(z, x)$ is a Caratheodory function.
- The potential term $\xi(z)(u(z))^{p-1}$ is indefinite, that is, $\xi(.)$ is sign-changing and this makes the differential operator (left-hand side of the (p_λ)) non-coercive.

In the reaction (right-hand side of (p_λ)), we have the competing effects of a "concave" ($(p-1)$-sublinear) term $c(u(z))^{\tau-1}$ ($c > 0,\ 1 < \tau < p$) and of a parametric perturbation which is "convex" ($(p-1)$-superlinear). So, problem (p_λ) is a generalized version of the classical "concave-convex problem".

Motivation and Literature

1994

The study of such problems started with the seminal work of Ambrosetti-Brezis-Gerami, 1994 , who considered semilinear Dirichlet equations driven by the Laplacian and with no potential term (that is, $\xi = 0$).

2000,2003

Their work was extended to Dirichlet equations driven by the p-Laplacian, by Garcia Azorero-Peral Alonso-Manfredi and by Guo-Zhang .

2000, 2020

More general differential operators and reactions were considered by Papageorgiou-Radulescu-Repovs (anisotropic p-Laplacian equations) and by Papageorgiou-Vetro ($(p, 2)$-equations). In both works, the problem has Dirichlet boundary conditions, there is no potential term (thus the operator is coercive) and the parameter λ multiplies the concave term.

2019

Only Marano-Marino-Papageorgiou , deal with a Dirichlet p-Laplacian equation with no potential term and a parametric convex (superlinear) term.

2023

This work was extended recently by Gasiniski-Papageorgiou-Zhang to problems driven by the Robin p-Laplacian and with a positive potential term (thus, the differential operator is coercive).

2023

Recently Bai-Papageorgiou-Zeng, studied nonparametric Robin problems driven by a similar nonhomogeneous differential operator as (p_λ) plus an indefinite potential term. The authors prove a multiplicity result producing solutions with sign information (positive, negative and nodal (sign-changing) solutions).

2023

Finally, we should also mention the recent relevant work of Papageorgiou-Radulescu-Zhang , which examines a nonlinear eigenvalue problem for the Robin p-Laplacian plus a positive potential term. They prove a bifurcation-type result but for large values of the parameter $\lambda > 0$.

Let $\hat{l} \in C^1(0,\infty)$ with $\hat{l}(t) > 0$ for all $t > 0$. We assume that there exist constants $c_1,\ c_2 > 0$ and $1 < s < p$ such that

$$0 < \hat{c} \leq \frac{t\hat{l}'(t)}{\hat{l}(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \hat{l}(t) \leq c_2(t^{s-1} + t^{p-1})$$

for all $t > 0$. The hypotheses on the potential function $\xi(.)$ and the boundary coefficient $\beta(.)$ are the following:
$\mathbf{H_0}$: $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1]$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $\xi \neq 0$ or $\beta \neq 0$.

$\hat{\mathbf{H}}$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ satisfies the following conditions:

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $\mathbb{R}^+ := (0, \infty)$ and $a_0(t)t \to 0^+$ as $t \to 0^+$ such that $\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} > -1$;

(ii) $|\nabla a(y)| \leq c_3 \frac{\hat{l}(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \backslash \{0\}$ for some $c_3 > 0$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\hat{l}(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \backslash \{0\}$ and all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)sds$ for all $t > 0$, then for some $q \in (\tau, p]$ we have $\lim_{t\to 0^+} \frac{qG_0(t)}{t^q} \leq c^*$ and $t \mapsto G_0(t^{1/q})$ is convex.

The hypotheses on the perturbation $f(z,x)$ are the following:
H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $f(z,0) = 0$ for almost every $z \in \Omega$, and satisfies the following conditions:

(i) $0 \leq f(z,x) \leq a(z)(1 + x^{r-1})$ for almost every $z \in \Omega$ and all $x \geq 0$, where $a \in L^{\infty}(\Omega)$ and $p < r < p^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \frac{f(z,x)}{x^p} = +\infty$ uniformly for almost every $z \in \Omega$;

(iii) there exists $\theta \in \left(\max\{1, (r-p)\frac{N}{p}\}, p^* \right)$ such that

$$0 < \beta \leq \liminf_{x\to\infty} \frac{f(z,x)x - pF(z,x)}{x^{\theta}} \text{ uniformly for almost every } z \in$$

(iv) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} = 0$ uniformly for almost every $z \in \Omega$;

(v) for every $s > 0$, there exists $\eta_s > 0$ such that $0 < \eta_s \leq f(z,x)$ for almost every $z \in \Omega$, all $x \geq s$ and for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost every $z \in \Omega$

$$x \to f(z,x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Main Results

Theorem

If hypotheses $\hat{H}, H_0, H$ hold, then there exists λ^* such that

(a) *for all $\lambda \in (0, \lambda^*)$, problem (p_λ) has at least two positive solutions*
$$u_0, \hat{u} \in intC_+;$$
(b) *for $\lambda = \lambda^*$, problem (p_λ) has at least one positive solution*
$$u^* \in intC_+;$$
(c) *for all $\lambda > \lambda^*$, problem (p_λ) has no positive solutions.*

(here $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$
$\mathrm{int}\, C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}$)

Using variational tools combined with suitable truncation and comparison techniques, we proved the existence and multiplicity result for the positive solutions of the problem which is global in $\lambda > 0$ (a bifurcation-type result but for small values of $\lambda > 0$).

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Thank you!