On the manifold-valued ROF model

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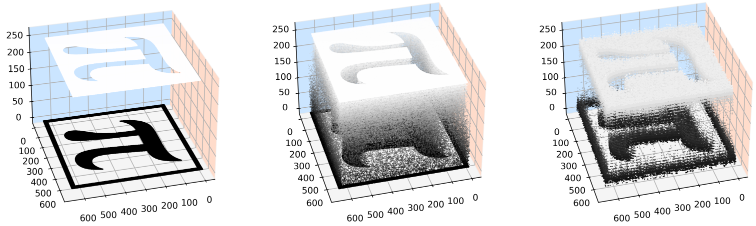
Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open and $f : \Omega \to [0,1]$ be the initial datum.
The *flat* ROF problem **[Rudin et al., 1992]** is defined as

$$\min_{u \in H^1(\Omega,[0,1])} \left(\underbrace{\int_\Omega |Du| dx + \frac{\lambda}{2} \int_\Omega |u-f|^2 dx + \frac{\sigma}{2} \int_\Omega |Du|^2 dx}_{E_{\lambda,\sigma}(u)} \right)$$



Introduction

Let (Σ, g) be compact surface, $(\mathcal{N}, h)$ be complete connected smooth n-dimensional Riemannian manifold and $f : \Sigma \to \mathcal{N}$ be the initial datum.

The *curved* ROF problem is defined as

$$\min_{u \in H^1(\Sigma,\mathcal{N})} \left(\underbrace{\int_\Omega |du| d\mu_g + \frac{\lambda}{2} \int_\Omega d_h^2(u,f) d\mu_g + \frac{\sigma}{2} \int_\Omega |du|^2 d\mu_g}_{E_{\lambda,\sigma}(u)} \right)$$

where

- d_h: geodesic distance on $\mathcal{N}$,
- $d\mu_g$: volume element associated with g
- $|du|_g^2 = \sum_{i=1}^N g^{\alpha\beta} \partial_\alpha u^i \partial_\beta u^i$,
- $u \in H^1(\Sigma, \mathcal{N})$: $u \in H^1(\Sigma; \mathbb{R}^N)$ such that $u(x) \in \mathcal{N}$ for a.e. $x \in \Sigma$

Goal: Study Lipschitz regularity of minimizer u given f Lipschitz.
Motivation: Diverse image supports, DS techniques in spherical/hyperbolic spaces, ...

Framework

According the E-L sequations, a minimizer of $E_{\lambda,\sigma}$ is solution of the system

$$\begin{cases} \pi_u \left(\operatorname{div}_g Z_u\right) = -\lambda \exp_u^{-1} f & \text{in } \Sigma, \\ \nu \cdot Z_u = 0 & \text{on } \partial\Sigma, \end{cases}$$

such that $\pi_u : \mathbb{R}^N \to T_u\mathcal{N}$ orthogonal projection and Z_u is defined as

$$Z_u := \left(\frac{1}{|du|_g} + \sigma\right) du.$$

$$\frac{du}{|du|_g} : x \longmapsto \begin{cases} \dfrac{du(x)}{|du(x)|_g}, & \text{if } du(x) \neq 0, \\ B_g(0,1) \subset T_x^*\Sigma \otimes T_{u(x)}\mathcal{N}, & \text{if } du(x) = 0. \end{cases}$$

Steps:

- Existence and uniqueness
- Regularity
  - Regularity of an approximate system
  - Lipschitz regularity

Existence and uniqueness

Let κ is the supremum of the sectional curvature over $\mathcal{N}$ and R_κ be the radius

$$R_\kappa := \frac{1}{2}\left\{\text{inj}(\mathcal{N}), \frac{\pi}{\sqrt{\kappa}}\right\}$$

Proposition

Let u be a minimizer. If $f(\Omega) \subset B_g(p,R)$ for some $p \in \mathcal{N}$ and $R < R_\kappa$, then $u(\Omega) \subset B_g(p,R)$.

Idea: Use Jost's 1-Lipschitz map trick **[Jost, 1983]**

Then, if $\Lambda := \{u \in H^1(\Omega) : u(\Omega) \subset B_g(p,R)\}$ and $f(\Omega) \subset B_g(p,R)$, we have

Proposition

For any $\lambda > 0$ and $\sigma \geq 0$, there exists a minimum of $E_{\lambda,\sigma}$ in Λ.

Idea: Use classical BV theory **[Ambrosio et al., 2000]**

Proposition

If $\kappa \leq 0$, $E_{\lambda,\sigma}$ has an unique minimizer in Λ.

Idea: Compute the 2nd variation along a geodesic (see **[Pigola and Veronelli, 2019]**)

Regularity

Let us consider the approximate system

$$\begin{cases} \pi_u\left(\operatorname{div}_g \mathcal{Z}_u\right) = -\lambda \exp_u^{-1} f & \text{in } \Sigma, \\ \nu \cdot \mathcal{Z}_u = 0 & \text{on } \partial\Sigma, \end{cases} \qquad (\mathcal{S}^f_{\varepsilon,\sigma})$$

such that $\mathcal{Z}$ is defined as

$$\mathcal{Z}_u := \left(\frac{1}{\sqrt{|du|_g^2 + \varepsilon^2}} + \sigma \right) du$$

Adapting **[Arkhipova, 1996]**, we get regularity up to the boundary:

Theorem

Let u be a weak solution of $(\mathcal{S}^f_{\varepsilon,\sigma})$. Then $u \in C^{0,\beta}(\overline{\Sigma};\mathcal{N})$ for all $\beta \in (0,1)$.

Theorem

Let u be a weak solution of $(\mathcal{S}^f_{\varepsilon,\sigma})$. Then $u \in C^{1,\beta_0}(\overline{\Sigma};\mathcal{N})$ for some $\beta_0 \in (0,1)$.

Technical tools: Poincaré inequality in H^1_Γ, Cacciopoli inequality, higher integrability **[Cabezas-Rivas et al., 2024]**

Regularity

By **[Karcher, 1977]**, we define $\{f_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\Sigma, \mathcal{N})$ such that $f_\varepsilon(p) \to f(p)$.

Proposition

Let u be a weak solution of $(\mathcal{S}^{f_\varepsilon}_{\varepsilon,\sigma})$. Then $u \in C^{3,\beta_0}(\overline{\Omega})$.

Idea: McShane's extension + direct method of CoV + bootstraping method

Proposition

Let u be a weak solution of $(\mathcal{S}^{f_\varepsilon}_{\varepsilon,\sigma})$ when $\kappa_\mathcal{N} \leq 0$. Then $u \in C^{0,1}(\Omega)$ and

$$\|\nabla u\|_\infty \leq \frac{C}{co_\kappa(R)} \|\nabla f\|_\infty.$$

Idea: (Bochner inequality + comparison thm) or **[Porretta, 2021]**

Current work: $\kappa_\mathcal{N} > 0$?

Proposition

Let u be the minimizer of $E_{\lambda,\sigma}$ when $\kappa_\mathcal{N} \leq 0$. Then $u \in C^{0,1}(\Omega)$.

Idea: Adapt **[Giacomelli et al., 2019]**

Thank you for your attention!

Děkuji za pozornost!

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