Towards mathematically justifying nonlinear constitutive relations between stress and linearized strain

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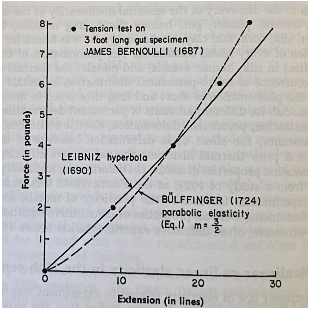

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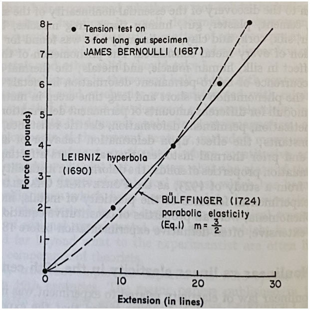

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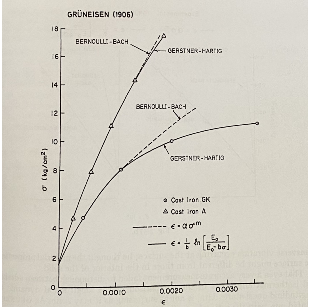

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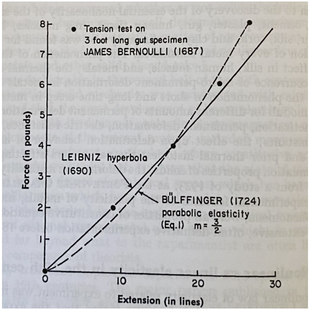

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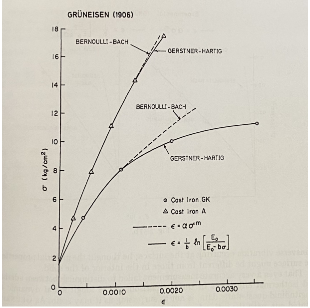

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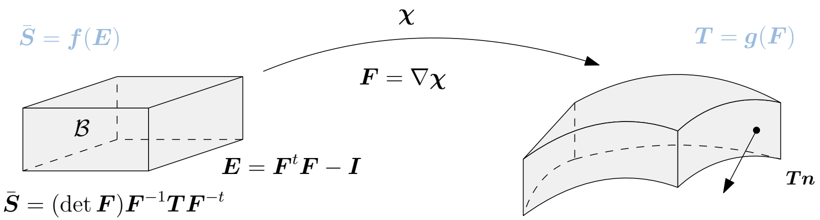
Goal: Propose a mathematical asymptotic framework where nonlinear constitutive relations between stress and linearized strain rigorously emerge as leading-order approximations to those describing finite elastic bodies.



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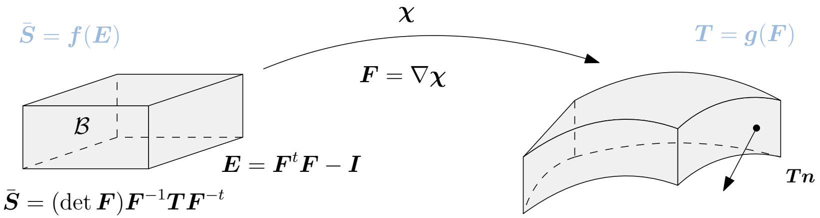

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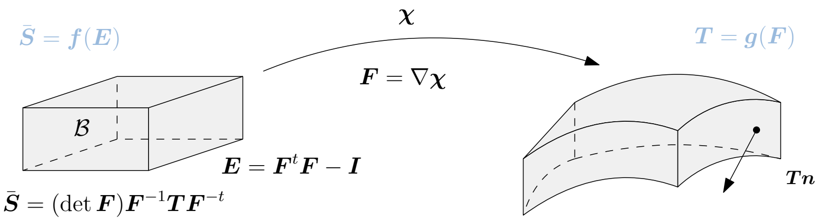

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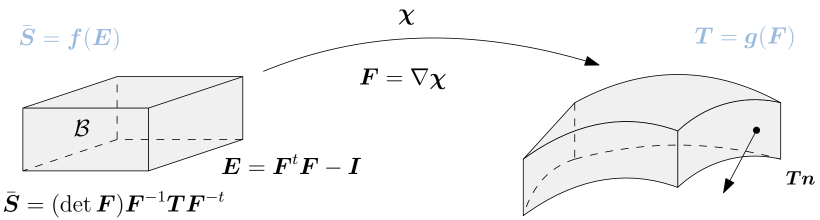
$$E = \epsilon + O(\delta_0^2), \quad \epsilon = \frac{1}{2}(F + F^t) - I = O(\delta_0),$$

$$\bar{S} = f_E(0)[\epsilon] + O(\delta_0^2).$$



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The response of a fixed material is described by the classical linearized elastic stress-strain relationship

$$\sigma = \mathsf{C}[\epsilon],$$

to leading order, as $\delta_0 \to 0$.

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Moral: By selecting δ_0 as the asymptotic parameter governing limiting behavior, a fixed constitutive relation is always approximated by a **linear** relation between stress and linearized strain, to leading order, as $\delta_0 \to 0$.

A different asymptotic parameter. Consider

$$E = \epsilon + \frac{1}{2}\epsilon^2, \quad \epsilon = -1 + (1 + 2E)^{1/2},$$
$$E = \delta a (1 + |a\bar{S}|^p)^{-1/p} \bar{S},$$

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Moral: If δ is the asymptotic parameter determining the limiting behavior for a family of constitutive relations, then these relations can be described by **nonlinear** relations between stress and linearized strain, to leading order, as $\delta \to 0$.

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$$\forall \boldsymbol{E}, \bar{\boldsymbol{S}}, \ |\boldsymbol{f}_\delta(\boldsymbol{E}, \bar{\boldsymbol{S}})| \leq C_0 \delta, \quad \forall \boldsymbol{E}_1 \neq \boldsymbol{E}_2, \bar{\boldsymbol{S}}, \ \frac{|\boldsymbol{f}_\delta(\boldsymbol{E}_2, \bar{\boldsymbol{S}}) - \boldsymbol{f}_\delta(\boldsymbol{E}_1, \bar{\boldsymbol{S}})|}{|\boldsymbol{E}_2 - \boldsymbol{E}_1|} \leq C_1,$$

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Example. $\boldsymbol{f}_\delta(\bar{\boldsymbol{S}}) = \delta a (1 + |a\bar{\boldsymbol{S}}|^p)^{-1/p} \bar{\boldsymbol{S}}$, or more generally,

$$\boldsymbol{f}_\delta(\boldsymbol{E}, \bar{\boldsymbol{S}}) = \delta \boldsymbol{f}_1\big(\boldsymbol{E}/\delta, \bar{\boldsymbol{S}}\big)$$

where $\boldsymbol{f}_1 : U \times V \to \mathrm{Sym}$ is a bounded Lipschitz continuous function.

Theorem (Rajagopal-R. 24)

Let $\boldsymbol{f}_\delta =: U_\delta \times V \to \mathrm{Sym}$ *be a family of strain-limiting functions with limiting small strains. Let* $\bar{\boldsymbol{S}} \in V$. *Assume that there exists* $r > 0$ *such that for each* $\delta > 0$ *sufficiently small, there exists* $\boldsymbol{E}_\delta \in U_\delta$ *such that* $B(\boldsymbol{E}_\delta, r\delta) \subseteq \delta U$ *and*

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$$\boldsymbol{C}_\delta := \boldsymbol{I} + 2\boldsymbol{E}_\delta, \quad \boldsymbol{F}_\delta := \boldsymbol{R}_\delta \boldsymbol{C}_\delta^{1/2}, \quad \boldsymbol{\epsilon}_\delta := \frac{1}{2}(\boldsymbol{F}_\delta + \boldsymbol{F}_\delta^T) - \boldsymbol{I},$$

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with $|\boldsymbol{R}_\delta - \boldsymbol{I}| < C_2 \delta$. Then for all δ sufficiently small, $(\boldsymbol{\epsilon}_\delta, \boldsymbol{\sigma}_\delta) \in U_\delta \times V$, and

$$\boldsymbol{\epsilon}_\delta = \boldsymbol{f}_\delta(\boldsymbol{\epsilon}_\delta, \boldsymbol{\sigma}_\delta) + \boldsymbol{O}(\delta^2), \quad \text{as } \delta \to 0,$$

where the big-oh term depends on C_0, C_1, C_2, and $D_0|\bar{\boldsymbol{S}}|$.

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By the **Theorem**, we can conclude that

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The leading order constitutive relation (and variants) have been studied by a number of authors recently including: Rajagopal 21', Itou-Kovtunenko-Rajagopal 21', Murru-Rajagopal 21', Murru et al 22', Prusa-Rajagopal-Wineman 22', Vajipeyajula-Murru-Rajagopal 23', and Jeyavel et al 24'.

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In particular, the **Theorem** rationalizes hyperelastic theories for infinitesimal displacement gradients that use non-quadratic stored energies of the linearized strain.

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- For a fixed external body force $\boldsymbol{b}$, Dirichlet conditions on the boundary, and for all δ sufficiently small, does solvability of the "linearized" equilibrium equations,

$$\mathbf{0} = \operatorname{Div} \boldsymbol{\sigma}_{L,\delta} + \boldsymbol{b}, \quad \boldsymbol{\sigma}^T_{L,\delta} = \boldsymbol{\sigma}_{L,\delta},$$
$$\boldsymbol{\epsilon}_{L,\delta} = \boldsymbol{f}_\delta(\boldsymbol{\epsilon}_{L,\delta}, \boldsymbol{\sigma}_{L,\delta}),$$

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  imply solvability of the fully nonlinear equilibrium equations,

$$\mathbf{0} = \operatorname{Div} \boldsymbol{S}_\delta + \boldsymbol{b}, \quad \boldsymbol{S}_\delta \boldsymbol{F}_\delta^T = \boldsymbol{F}_\delta \boldsymbol{S}_\delta^T,$$
$$\boldsymbol{S}_\delta = \boldsymbol{F}_\delta \bar{\boldsymbol{S}}, \quad \boldsymbol{E}_\delta = \boldsymbol{f}_\delta(\boldsymbol{E}_\delta, \bar{\boldsymbol{S}}_\delta),$$

Open questions.

- Extend this framework to nonlinear viscoelastic constitutive relations between stress and linearized strain.
- Extend this framework to elastic or viscoelastic rods and shells.
- For a fixed external body force $\boldsymbol{b}$, Dirichlet conditions on the boundary, and for all δ sufficiently small, does solvability of the "linearized" equilibrium equations,

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imply solvability of the fully nonlinear equilibrium equations,

$$\mathbf{0} = \mathrm{Div}\,\boldsymbol{S}_\delta + \boldsymbol{b}, \quad \boldsymbol{S}_\delta \boldsymbol{F}_\delta^T = \boldsymbol{F}_\delta \boldsymbol{S}_\delta^T,$$
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with

$$\boldsymbol{E}_\delta = \boldsymbol{\epsilon}_{L,\delta} + \boldsymbol{O}(\delta^2), \quad \bar{\boldsymbol{S}}_\delta = \boldsymbol{\sigma}_{L,\delta} + \boldsymbol{O}(\delta), \quad \boldsymbol{S}_\delta = \boldsymbol{\sigma}_{L,\delta} + \boldsymbol{O}(\delta),$$

as $\delta \to 0$?

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as $\delta \to 0$? Analogous results are known to hold for classical linearized elasticity (as $\delta_0 \to 0$), see, e.g., Stoppolli 54-55'.

Thank you for your attention!