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Finite element methods for magnetoelastic materials

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joint work with

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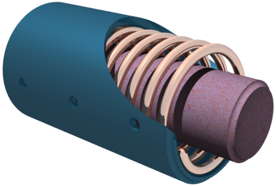
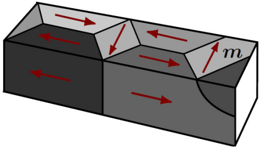
What are magnetoelastic materials?

- smart materials with strong interplay between mechanical and magnetic properties
- also called magnetostrictive
- very small effect
  - Co-Fe-Ni alloys ⇝ 60 ppm
  - 'giant' magnetostrictive materials (e.g., Terfenol-D) ⇝ 1000-2000 ppm
  - MSMAs ⇝ 6%



@ Cut-away of a transducer (source: Wikipedia)

Magnetoelastic coupling



- magnetic energy (exchange only)

$$\mathcal{E}_{\mathrm{mag}}[\boldsymbol{m}] = \frac{1}{2}\int_\Omega |\boldsymbol{\nabla}\boldsymbol{m}|^2$$

  - magnetization $\rightsquigarrow \boldsymbol{m} : \Omega \subset \mathbb{R}^3 \to \mathbb{S}^2$

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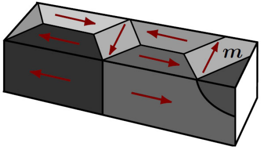

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  - magnetization $\rightsquigarrow$ $\boldsymbol{m} : \Omega \subset \mathbb{R}^3 \to \mathbb{S}^2$

- elastic energy ($\mathbb{C} \in \mathbb{R}^{3^4}$ fully symmetric and positive definite)

$$\mathcal{E}_{\mathrm{el}}[\boldsymbol{u}, \boldsymbol{m}] = \frac{1}{2}\int_\Omega [\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m})] : \{\mathbb{C} : [\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m})]\} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}$$

  - displacement $\rightsquigarrow$ $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ satisfying $\boldsymbol{u} = \boldsymbol{0}$ on Γ_D
  - total strain $\rightsquigarrow$ $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top\right)$
  - magnetostrain ($\mathbb{Z} \in \mathbb{R}^{3^4}$ minorly symmetric) $\rightsquigarrow$ $\boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m}) = \mathbb{Z}(\boldsymbol{m} \otimes \boldsymbol{m})$
  - volume and surface forces $\rightsquigarrow$ $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g} : \Gamma_N \to \mathbb{R}^3$

Total energy & dynamics

- total energy

$$
\begin{aligned}
\mathcal{E}[\boldsymbol{u}, \boldsymbol{m}] &= \mathcal{E}_{\mathrm{mag}}[\boldsymbol{m}] + \mathcal{E}_{\mathrm{el}}[\boldsymbol{u}, \boldsymbol{m}] \\
&= \frac{1}{2}\int_\Omega |\boldsymbol{\nabla} \boldsymbol{m}|^2 + \frac{1}{2}\int_\Omega [\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m})] : \{\mathbb{C} : [\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m})]\} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}
\end{aligned}
$$

Total energy & dynamics

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- stress $\rightsquigarrow$ $\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{m}) = \mathbb{C} : [\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m})]$

- effective field $\rightsquigarrow$ $\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{u},\boldsymbol{m}] = -\dfrac{\delta\mathcal{E}[\boldsymbol{u},\boldsymbol{m}]}{\delta\boldsymbol{m}} = \Delta\boldsymbol{m} + 2\,[\mathbb{Z}^\top : \boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{m})]\boldsymbol{m} = \Delta\boldsymbol{m} + \boldsymbol{h}_{\mathrm{m}}[\boldsymbol{u},\boldsymbol{m}]$

Total energy & dynamics

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- coupled system of PDEs (conservation of momentum + Landau–Lifshitz–Gilbert equation)

$$\begin{aligned} \partial_{tt}\boldsymbol{u} &= \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{m}) + \boldsymbol{f} && \text{in } \Omega\times(0,T) \\ \partial_t\boldsymbol{m} &= -\boldsymbol{m}\times\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{u},\boldsymbol{m}] + \alpha\,\boldsymbol{m}\times\partial_t\boldsymbol{m} && \text{in } \Omega\times(0,T) \end{aligned}$$

Literature

- existence of weak solutions
- convergent integrator toward strong solution
- convergent integrator toward weak solution

- Visintin: Japan J. Appl. Math. 2 (1985)
- Carbou, Efendiev, Fabrie: Math. Meth. Appl. Sci. 34 (2011)
- Banas: Math. Meth. Appl. Sci. 28 (2005), J. Comp. Appl. Math. 215 (2008)
- Banas, Page, Praetorius, Rochat: IMA J. Numer. Anal. 34 (2014)

Challenges and main aim

- nonlinearities
- nonuniqueness and low regularity of weak solutions
- nonconvex pointwise unit-length constraint $\rightsquigarrow$ $|\boldsymbol{m}| = 1$
- energy law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}[\boldsymbol{u},\boldsymbol{m}] + \frac{1}{2}\int_\Omega |\partial_t \boldsymbol{u}|^2\right) = -\alpha \int_\Omega |\partial_t \boldsymbol{m}|^2 \leq 0$$

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Aim

- develop fully discrete numerical scheme
  - unconditionally convergent
  - structure-preserving
  - stable without assuming weakly acute meshes

Decoupled algorithm (1/3)

Main structure

- time discretization $\rightsquigarrow$ $0 = t_0 < t_1 < t_2 < \dots$ with $t_i = ik$ $\rightsquigarrow$ time-step size $k > 0$
- spatial discretization $\rightsquigarrow$ $\mathcal{T}_h$ tetrahedral mesh of $\Omega \subset \mathbb{R}^3$ $\rightsquigarrow$ mesh size $h > 0$ $\rightsquigarrow$ P1-FEM

Banas, Page, Praetorius, Rochat: IMA J. Numer. Anal. 34 (2014)

Normington, Ruggeri: arXiv:2309.00605

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- decoupled algorithm
  - input
    - initial approximations $\boldsymbol{u}_h^0 \approx \boldsymbol{u}^0$ and $\boldsymbol{m}_h^0 \approx \boldsymbol{m}^0$
  - loop
    - magnetization update: Use $\boldsymbol{u}_h^i \approx \boldsymbol{u}(t_i)$ and $\boldsymbol{m}_h^i \approx \boldsymbol{m}(t_i)$ to compute $\boldsymbol{m}_h^{i+1} \approx \boldsymbol{m}(t_{i+1})$
    - displacement update: Use $\boldsymbol{u}_h^i \approx \boldsymbol{u}(t_i)$ and $\boldsymbol{m}_h^{i+1} \approx \boldsymbol{m}(t_{i+1})$ to compute $\boldsymbol{u}_h^{i+1} \approx \boldsymbol{u}(t_{i+1})$
  - output
    - sequence of approximations $\{\boldsymbol{u}_h^i\}$ and $\{\boldsymbol{m}_h^i\}$

Banas, Page, Praetorius, Rochat: IMA J. Numer. Anal. 34 (2014)

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Decoupled algorithm (2/3)

- equivalent reformulation of LLG for linear velocity $\boldsymbol{v} = \partial_t \boldsymbol{m}$

$$\alpha \boldsymbol{v} + \boldsymbol{m} \times \boldsymbol{v} = \boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{u}, \boldsymbol{m}] - (\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{u}, \boldsymbol{m}] \cdot \boldsymbol{m})\boldsymbol{m}$$

- magnetization update
  (i) linear constrained variational formulation posed on discrete tangent space $\rightsquigarrow \boldsymbol{v}_h^i$

$$\alpha \boldsymbol{v}_h^i + \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i - k\,\Delta_h \boldsymbol{v}_h^i = \Delta_h \boldsymbol{m}_h^i + \boldsymbol{h}_{\mathrm{m}}[\boldsymbol{u}_h^i, \Pi_h \boldsymbol{m}_h^i]$$

  (ii) linear time-stepping

$$\boldsymbol{m}_h^{i+1} := \boldsymbol{m}_h^i + k\,\boldsymbol{v}_h^i$$

Alouges: Discrete Contin. Dyn. Syst. Ser. S 1 (2008)
Bartels: Math. Comp. 85 (2016)
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  - (i) linear constrained variational formulation posed on discrete tangent space $\rightsquigarrow \boldsymbol{v}_h^i$

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$$\boldsymbol{m}_h^{i+1} := \boldsymbol{m}_h^i + k\,\boldsymbol{v}_h^i$$

- properties
  - fully linear
  - implicit treatment of exchange field
  - explicit treatment of (displacement-dependent) magnetoelastic field + nodal projection Π_h

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Decoupled algorithm (3/3)

- conservation of momentum

$$\partial_{tt}\boldsymbol{u} - \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} - \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}_{\mathrm{m}}(\boldsymbol{m}))$$

- displacement update

(iii) linear variational formulation $\rightsquigarrow \boldsymbol{u}_h^{i+1}$

$$\frac{\boldsymbol{u}_h^{i+1} - 2\boldsymbol{u}_h^{i} + \boldsymbol{u}_h^{i-1}}{k^2} - \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{i+1})) = \boldsymbol{f} - \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}_{\mathrm{m}}(\Pi_h\boldsymbol{m}_h^{i+1}))$$

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Decoupled algorithm (3/3)

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- properties
  - fully linear
  - nodal projection Π_h on magnetization

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Results

- algorithm is unconditionally well-posed
- continuous energy law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}[\boldsymbol{u},\boldsymbol{m}] + \frac{1}{2}\int_\Omega |\partial_t \boldsymbol{u}|^2\right) = -\alpha \int_\Omega |\partial_t \boldsymbol{m}|^2 \leq 0$$

- discrete energy law

$$\mathcal{E}[\boldsymbol{u}_h^{i+1},\boldsymbol{m}_h^{i+1}] + \frac{1}{2}\left\|d_t \boldsymbol{u}_h^{i+1}\right\|^2 - \mathcal{E}[\boldsymbol{u}_h^{i},\boldsymbol{m}_h^{i}] - \frac{1}{2}\left\|d_t \boldsymbol{u}_h^{i}\right\|^2 = -\alpha k \left\|\boldsymbol{v}_h^i\right\|_h^2 - D_{h,k}^i - E_{h,k}^i$$

  - $D_{h,k}^i \geq 0 \rightsquigarrow$ artificial dissipation
  - $E_{h,k}^i \rightsquigarrow$ error (linearization, decoupling, nodal projection)
- unconditional stability (no need of weakly acute meshes!)
- control of constraint violation

$$\left\|I_h\left[|\boldsymbol{m}_h^j|^2\right] - 1\right\|_{L^1(\Omega)} \leq Ck$$

Normington, Ruggeri: arXiv:2309.00605

Unconditional convergence

- $\boldsymbol{u}_{hk}, \boldsymbol{m}_{hk} \rightsquigarrow$ piecewise affine and globally continuous time reconstructions

Theorem

- *convergence of initial data approximations*

$\Longrightarrow$ *there exist weak solution* $(\boldsymbol{u}, \boldsymbol{m})$ *and (nonrelabeled) subsequences of* $\{\boldsymbol{u}_{hk}\}$ *and* $\{\boldsymbol{m}_{hk}\}$ *s.t.*

$$
\begin{aligned}
\boldsymbol{u}_{hk} &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{in } L^\infty(0,T;\boldsymbol{H}^1_D(\Omega)) \\
\partial_t \boldsymbol{u}_{hk} &\overset{*}{\rightharpoonup} \partial_t \boldsymbol{u} && \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \\
\boldsymbol{m}_{hk} &\overset{*}{\rightharpoonup} \boldsymbol{m} && \text{in } L^\infty(0,T;\boldsymbol{H}^1(\Omega;\mathbb{S}^2)) \\
\partial_t \boldsymbol{m}_{hk} &\rightharpoonup \partial_t \boldsymbol{m} && \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega))
\end{aligned}
$$

as $h, k \longrightarrow 0$

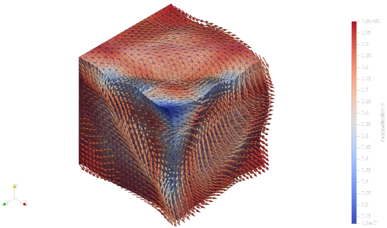
- constructive proof

Normington, Ruggeri: arXiv:2309.00605

Summary & Outlook

Summary

- small strain model of magnetoelastic materials
- fully discrete structure-preserving numerical scheme
- well-posedness, stability & convergence results
- numerical experiments (not discussed)



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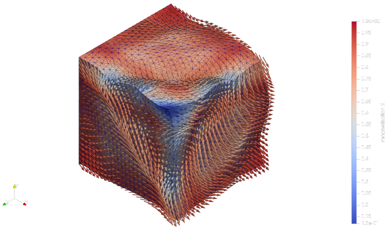

Outlook

- extension to nonsimple materials (strain gradient elasticity)
- extension to finite strain magnetoelasticity

Normington, Ruggeri: arXiv:2309.00605

Thank you for your attention!

Hywel Normington, Michele Ruggeri
A decoupled, convergent and fully linear algorithm for the Landau–Lifshitz–Gilbert equation with magnetoelastic effects
arXiv:2309.00605

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Finite strain magnetoelasticity

- $\boldsymbol{y} : \Omega \longrightarrow \mathbb{R}^3 \rightsquigarrow$ deformation
- $\widetilde{\boldsymbol{m}} : \boldsymbol{y}(\Omega) \longrightarrow \mathbb{R}^3 \rightsquigarrow$ magnetization
- minimize

$$\mathcal{E}[\boldsymbol{y}, \widetilde{\boldsymbol{m}}] = \int_\Omega W(\boldsymbol{\nabla} \boldsymbol{y}(x), \widetilde{\boldsymbol{m}} \circ \boldsymbol{y}(x))\, \mathrm{d}x + \frac{1}{2} \int_{\boldsymbol{y}(\Omega)} |\boldsymbol{\nabla} \widetilde{\boldsymbol{m}}(y)|^2\, \mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^3} \left|\nabla \widetilde{\phi}(y)\right|^2 \mathrm{d}y$$

subject to

$$|\widetilde{\boldsymbol{m}} \circ \boldsymbol{y}| \det \boldsymbol{\nabla} \boldsymbol{y} = 1$$

where

$$\Delta \widetilde{\phi} = \nabla \cdot (\chi_{\boldsymbol{y}(\Omega)} \widetilde{\boldsymbol{m}}) \quad \text{in } \mathbb{R}^3$$

Bresciani, Davoli, Kružík: Ann. Inst. H. Poincaré Anal. Non Linéaire 40 (2023)

Kružík, Normington, Ruggeri: In preparation