Regularity of surfaces with nearly minimal bending

joint work with C. Scharrer (Bonn)

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Total bending of elastic membranes

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- measure the **total bending** of a surface $\Sigma^2 \subset \mathbb{R}^3$ by the **Willmore energy**

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$$\min \quad \int_\Sigma \left(\frac{\beta}{2}(H - H_0)^2 + \gamma K\right) \mathrm{d}\mu$$

subject to $\mathrm{Area}(\Sigma) = a, \mathrm{Vol}(\Sigma) = v$

A red blood cell.[a]

[a] Database Center for Life Science (DBCLS) at commons.wikimedia.org

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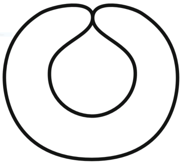
- [Li–Yau '82]: if Σ is smooth and $\boldsymbol{\mathcal{W}(\Sigma) < 8\pi}$, then Σ is **embedded**



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Let μ be an integral 2-varifold with finite mass and $\vec{H} \in L^2$. If $\mathcal{W}(\mu) < \mathbf{6\pi}$, then μ can be parametrized by a **conformal $W^{2,2}$-Lipschitz** embedding.

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THANK YOU FOR YOUR ATTENTION!