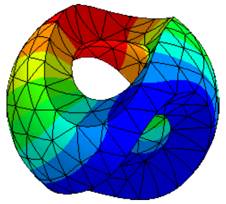
Distributional finite elements with applications for elasticity, fluids and curvature

Joachim Schöberl

Computational Mathematics in Engineering
Institute of Analysis and Scientific Computing
TU Wien



based on joint work with

Jay Gopalakrishnan (PSU), Kaibo Hu (University of Edinburgh),
Astrid Pechstein (JKU Linz), Philip Lederer (University of Hamburg),
Michael Neunteufel (PSU), Edoardo Bonetti (TU Wien)

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Vector-valued function spaces

$$\begin{aligned} H(\mathrm{curl}) &= \{v \in L_2(\mathbb{V}) : \mathrm{curl}\, u \in L_2(\mathbb{V})\} \\ H(\mathrm{div}) &= \{v \in L_2(\mathbb{V}) : \mathrm{div}\, u \in L_2\} \end{aligned}$$

- de Rham sequence:

$$H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

- regular decomposition $H(\mathrm{curl}) = [H^1]^3 + \nabla H^1$, $H(\mathrm{div}) = [H^1]^3 + \mathrm{curl}[H^1]^3$
- dual space:

$$\begin{aligned} \|u\|_{H(\mathrm{curl})^*} &= \sup_{v \in H(\mathrm{curl})} \frac{\langle u, v\rangle}{\|v\|_{H(\mathrm{curl})}} = \sup_{v \in H(\mathrm{curl})} \frac{\langle u, v\rangle}{\inf_{v = \nabla\phi + z} \|\phi\|_{H^1} + \|z\|_{H^1}} = \sup_{\phi, z} \frac{\langle u, \nabla\phi + z\rangle}{\|\phi\|_{H^1} + \|z\|_{H^1}} \\ &\simeq \sup_{\phi} \frac{\langle \mathrm{div}\, u, \phi\rangle}{\|\phi\|_{H^1}} + \sup_{z} \frac{\langle u, z\rangle}{\|\phi\|_{H^1}} = \|u\|_{H^{-1}} + \|\,\mathrm{div}\, u\|_{H^{-1}} =: \|u\|_{H^{-1}(\mathrm{div})} \end{aligned}$$

- tangential / normal boundary traces $\Rightarrow$ tangential / normal continuous finite elements ($\mathcal{N}^k, \mathcal{BDM}^k$)

Matrix-valued function spaces

$$\begin{aligned}
\widehat{H}(dd) := \widehat{H}(\operatorname{div}\operatorname{div}) &:= \{\sigma \in H^{-1}(\mathbb{S}) : \operatorname{div}\sigma \in H^{-1}(\mathbb{V}), \operatorname{div}\operatorname{div}\,\sigma \in H^{-1}(\mathbb{R})\} \\
\widehat{H}(cd) := \widehat{H}(\operatorname{curl}\operatorname{div}) &:= \{\sigma \in H^{-1}(\mathbb{T}) : \operatorname{div}\sigma \in H^{-1}(\mathbb{V}), \operatorname{sym-curl}\sigma^T \in H^{-1}(\mathbb{S}), \operatorname{curl}\operatorname{div}\sigma \in H^{-1}(\mathbb{V})\} \\
\widehat{H}(cc) := \widehat{H}(\operatorname{curl}\operatorname{curl}) &:= \{\sigma \in H^{-1}(\mathbb{S}) : \operatorname{curl}\sigma \in H^{-1}(\mathbb{T}), \operatorname{curl}^T\operatorname{curl}\sigma \in H^{-1}(\mathbb{S})\}
\end{aligned}$$

regular sub-spaces:

$$\begin{aligned}
\widetilde{H}(dd) := &:= \{\sigma \in L_2(\mathbb{S}) : \operatorname{div}\operatorname{div}\,\sigma \in H^{-1}(\mathbb{R})\} \\
\widetilde{H}(cd) &:= \{\sigma \in L_2(\mathbb{T}) : \operatorname{curl}\operatorname{div}\sigma \in H^{-1}(\mathbb{V})\} \\
\widetilde{H}(cc) &:= \{\sigma \in L_2(\mathbb{S}) : \operatorname{curl}^T\operatorname{curl}\sigma \in H^{-1}(\mathbb{S})\}
\end{aligned}$$

with $\mathbb{V} = \mathbb{R}^3$, $\mathbb{S} = \{\sigma \in \mathbb{R}^{3\times 3} : \sigma = \sigma^T\}$, $\mathbb{T} = \{\eta \in \mathbb{R}^{3\times 3} : \operatorname{tr}\eta = 0\}$

Finite element spaces

finite element spaces for $k \geq 0$

$$\begin{aligned} V_{dd}^k &= \{\sigma \in L_2(\mathbb{S}) : \sigma_{|T} \in P^k(\mathbb{S}), \sigma_{nn} \text{ continuous}\} \\ V_{cd}^k &= \{\sigma \in L_2(\mathbb{T}) : \sigma_{|T} \in P^k(\mathbb{T}), \sigma_{nt} \text{ continuous}\} \\ V_{cc}^k &= \{\sigma \in L_2(\mathbb{S}) : \sigma_{|T} \in P^k(\mathbb{S}), \sigma_{tt} \text{ continuous}\} \end{aligned}$$

- In 2D, V_{dd}^k is the Hellan-Herrmann-Johnson finite element space, used for Kirchhoff plates.
  In 3D: Phd thesis Astrid Pechstein, context of TDNNS method for elasticity.

- $H(\operatorname{curl}\operatorname{div})$ space and V_{cd} finite elements in Phd thesis Philip Lederer with applications for Stokes (MCS method), P. Lederer, J. Gopalakrishnan + JS [20,20]

- V_{cc} is the Regge finite element space [Christiansen '11, Li '18]

- Shape functions are defined on reference elements, two-sided Piola/covariant transformations preserve normal/tangential components

- mapping to manifold meshes (shells, intrinsic curvature), shape derivatives

Hypercomplex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\text{grad}} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\downarrow{\scriptstyle \text{grad}} & & \downarrow{\scriptstyle \text{def}} & & \downarrow{\scriptstyle \text{dev}\,\text{grad}^T} & & \downarrow{\scriptstyle \text{grad}} \\
H(\text{curl}) & \xrightarrow{\text{def}} & H_{cc}(\mathbb{S}) & \xrightarrow{\text{curl}} & H_{cd}(\mathbb{T}) & \xrightarrow{\text{div}} & H^{-1}(\text{curl}) \\
\downarrow{\scriptstyle \text{curl}} & & \downarrow{\scriptstyle \text{curl}^T} & & \downarrow{\scriptstyle \text{sym}\,\text{curl}^T} & & \downarrow{\scriptstyle \text{curl}} \\
H(\text{div}) & \xrightarrow{\text{dev}\,\text{grad}} & H_{dc}(\mathbb{T}) & \xrightarrow{\text{sym}\,\text{curl}} & H_{dd}(\mathbb{S}) & \xrightarrow{\text{div}} & H^{-1}(\text{div}) \\
\downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} \\
L^2 & \xrightarrow{\text{grad}} & H^{-1}(\text{curl}) & \xrightarrow{\text{curl}} & H^{-1}(\text{div}) & \xrightarrow{\text{div}} & H^{-1}
\end{array}
$$

with $\operatorname{def} u = \frac{1}{2}(\nabla u + \nabla u^T)$, $\operatorname{dev}\eta = \eta - \frac{1}{3}\operatorname{tr}\eta I$, $\operatorname{curl}^T \gamma = (\operatorname{curl}\gamma^T)^T$

dual spaces are at opposite positions:

$$[V^{k,l}]^* = V^{3-k,3-l}$$

related to BGG complex [Doug Arnold, Kaibo Hu]

Three exact sequences with 2^{nd}-order operators

from Arnold+Hu:

- Hessian complex (using $\operatorname{div}\operatorname{sym-curl} = \frac{1}{2}\operatorname{curl}\operatorname{div}$):

$$H^1 \xrightarrow{\text{hess}} H(cc) \xrightarrow{\text{curl}} H(cd) \xrightarrow{\text{div}} H(\text{div})^*$$

- Elasticity complex

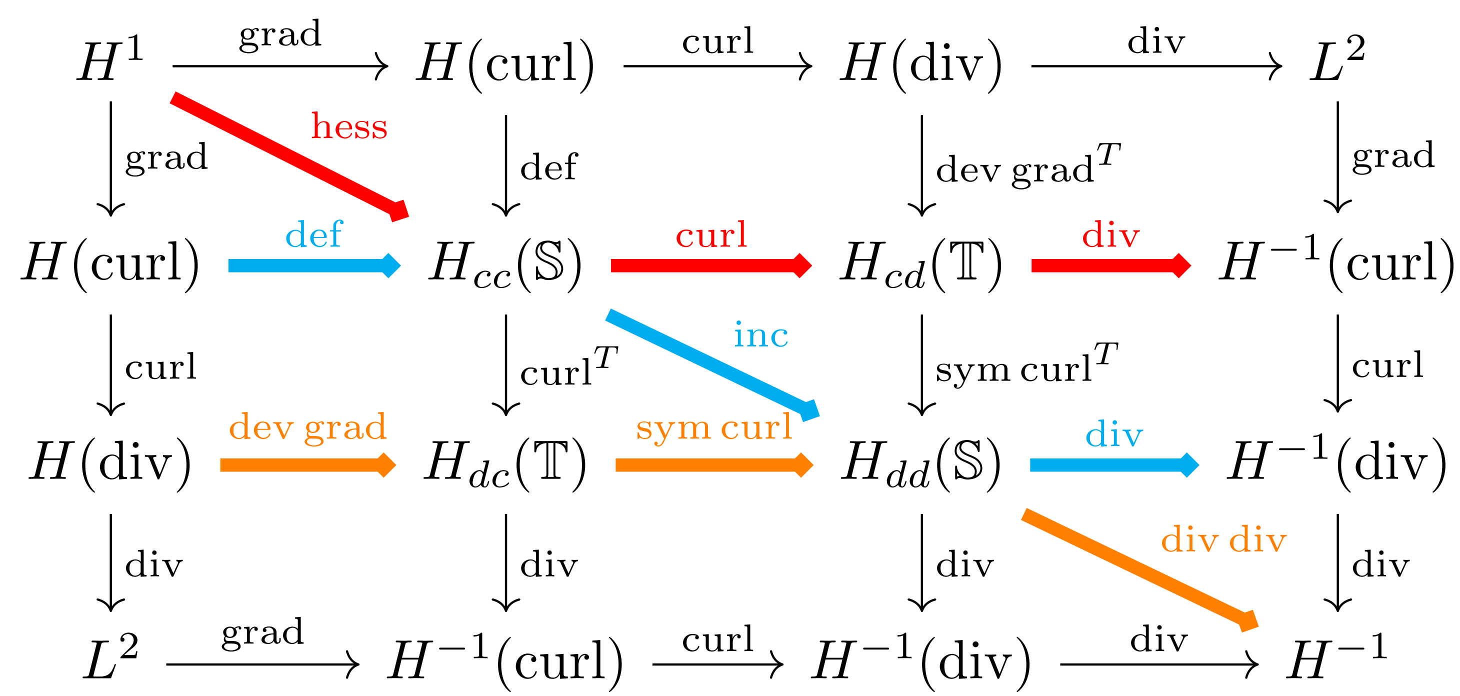
$$H(\text{curl}) \xrightarrow{\text{def}} H(cc) \xrightarrow{\text{inc}} H(dd) \xrightarrow{\text{div}} H(\text{curl})^*$$

- $\operatorname{div}\operatorname{div}$ complex (using $\operatorname{curl}\operatorname{sym-grad} = \frac{1}{2}\operatorname{dev}\operatorname{grad}\operatorname{curl}$):

$$H(\text{curl}) \stackrel{\text{dev grad}}{\longrightarrow} H(cd) \stackrel{\text{sym-curl}^T}{\longrightarrow} H(dd) \stackrel{\text{div div}}{\longrightarrow} H^{-1}$$

with $\text{hess} = \text{def}\,\text{grad}$ and $\text{inc} = \text{sym} - \text{curl}^T\,\text{curl}$

Exact sequences in the hypercomplex



Stability

Lemma: Continuity

$$\forall\, u \in H(\mathrm{curl})\ \forall\, \sigma \in \widehat{H}(dd): \quad \langle \operatorname{div} \sigma, v\rangle \leq \|\sigma\|_{\widehat{H}(dd)} \|u\|_{H(\mathrm{curl})}$$

Proof: Regular decomposition: $u = z + \nabla \varphi$ with $\|z\|_{H^1} + \|\varphi\|_{H^1} \leq \|u\|_{H(\mathrm{curl})}$.

Then

$$\begin{aligned}
\langle \operatorname{div} \sigma, u\rangle &= \langle \operatorname{div} \sigma, z + \nabla\varphi\rangle = \langle \operatorname{div} \sigma, z\rangle - \langle \operatorname{div} \operatorname{div} \sigma, \varphi\rangle \\
&\leq \|\operatorname{div} \sigma\|_{-1} \|z\|_1 + \|\operatorname{div} \operatorname{div} \sigma\|_{-1} \|\varphi\|_1 \leq \|\sigma\|_{\widehat{H}(dd)} \|u\|_{H(\mathrm{curl})}
\end{aligned}$$

Lemma: $\inf - \sup$ condition:

$$\forall\, u \in H(\mathrm{curl})\ \exists\, \sigma \in \widetilde{H}(dd): \quad \frac{\langle \operatorname{div} \sigma, u\rangle}{\|\sigma\|_{\widetilde{H}(dd)} \|u\|_{H(\mathrm{curl})}} \geq \beta$$

Proof: Given $u \in H(\mathrm{curl})$, solve elasticity problem: $(\varepsilon(w), \varepsilon(v)) = (u, v)_{H(\mathrm{curl})}$ and set $\sigma := \varepsilon(w)$.

Then $\langle \operatorname{div} \sigma, u\rangle = \|u\|^2_{H(\mathrm{curl})}$, and $\|\sigma\|_{L_2} \leq \|u\|_{H(\mathrm{curl})}$, $\|\operatorname{div} \operatorname{div} \sigma\|_{H^{-1}} \leq \|u\|_{L_2}$

Distributional derivatives

Let $\sigma \in V_{dd}^k$. Then the distributional divergence $f := \operatorname{div} \sigma$ is

$$\begin{aligned} \langle f, \varphi \rangle &= -\int \sigma : \nabla \varphi = -\sum_T \int_T \sigma : \nabla \varphi = \sum_T \int_T \operatorname{div} \sigma - \int_{\partial T} \sigma_n \varphi \\ &= \sum_T \int_T \operatorname{div} \sigma \varphi - \sum_E \int_E [\sigma_n] \varphi = \sum_T \int_T \underbrace{\operatorname{div} \sigma}_{f_T} \varphi - \sum_E \int_E \underbrace{[\sigma_{nt}]}_{f_E} \varphi_t \end{aligned}$$

$f = \operatorname{div} \sigma$ consists of element-terms and facet-terms:

$$\begin{aligned} f_T &= \operatorname{div}_T \sigma \\ f_E &= [\sigma_{nt}] \qquad \text{vector in tangential space} \end{aligned}$$

It can be applied to $v_h \in \mathcal{N} \subset H(\mathrm{curl})$.

Write duality pairing as

$$\langle \operatorname{div} \sigma, v \rangle \qquad \text{for } \sigma \in V_{dd}^k,\ v \in \mathcal{N}^k$$

Second distributional derivatives

Let f as above, and $g = \operatorname{div} f$. Then

$$
\begin{aligned}
\langle g, \varphi \rangle &= -\sum_T \int_T f_T \nabla \varphi - \sum_E \int_E f_E \nabla_t \varphi \\
&= \sum_T \int_T \operatorname{div}_T f_T \, \varphi + \sum_E \int_E ([f_{T,n}] + \operatorname{div}_t f_E) \, \varphi + \sum_V \sum_{T:V \in T} (\sigma_{n_1 t_1} - \sigma_{n_2 t_2}) \varphi
\end{aligned}
$$

$$
\begin{aligned}
g_T &= \operatorname{div}_T f_T \\
g_E &= [f_{T,n}] + \operatorname{div}_t f_E \\
g_V &= \sum_{T:V \in T} (\sigma_{n_1 t_1} - \sigma_{n_2 t_2})
\end{aligned}
$$

g is a measure and can be applied to $v_h \in \mathcal{L}^{k+1} \subset H^1$. Due to the arising point functionals, V_{dd} is slightly non-conforming for $H(dd)$.

Distributional differential operators

- $\langle \operatorname{hess} w, \sigma \rangle$
- $\langle \varepsilon(u), \sigma \rangle$, $\left\langle \operatorname{curl} \varepsilon(u), \eta^T \right\rangle$
- $\left\langle \operatorname{curl} \gamma, \eta^T \right\rangle$, $\langle \operatorname{inc} \gamma, \tilde{\gamma} \rangle$
- $\langle \text{sym-curl}\, \eta, \gamma \rangle$, $\langle \operatorname{div} \eta, q \rangle$
- $\langle \operatorname{div} \sigma, u \rangle, \langle \operatorname{div} \operatorname{div} \sigma, w \rangle$

with $w \in \mathcal{L}_k \subset H^1$, $u \in \mathcal{N}_k \subset H(\text{curl})$, $q \in \mathcal{BDM}_k \subset H(\text{div})$

$\gamma \in V_{cc}^k \subset \widetilde{H}(cc)$, $\eta \in V_{cd}^k \subset \widetilde{H}(cd)$, $\sigma \in V_{dd}^k \subset \widetilde{H}(dd)$.

Lowest order finite element hypercomplex

$$
\begin{array}{ccccccc}
\mathcal{L}_1 & \xrightarrow{\text{grad}} & \mathcal{N}_0 & \xrightarrow{\text{curl}} & \mathcal{R}T_0 & \xrightarrow{\text{div}} & P^0 \\
\Big\downarrow{\scriptstyle \text{grad}} & & \Big\downarrow{\scriptstyle \text{def}} & & \Big\downarrow{\scriptstyle \text{dev grad}^T} & & \Big\downarrow{\scriptstyle \text{grad}} \\
\mathcal{N}_0 & \xrightarrow{\text{def}} & V_{cc} & \xrightarrow{\text{curl}} & V_{cd} & \xrightarrow{\text{div}} & \delta_n^F \\
\Big\downarrow{\scriptstyle \text{curl}} & & \Big\downarrow{\scriptstyle \text{curl}^T} & & \Big\downarrow{\scriptstyle \text{sym curl}^T} & & \Big\downarrow{\scriptstyle \text{curl}} \\
\mathcal{R}T_0 & \xrightarrow{\text{dev grad}} & V_{cd}^T & \xrightarrow{\text{sym curl}} & V_{dd} & \xrightarrow{\text{div}} & \delta_t^E \\
\Big\downarrow{\scriptstyle \text{div}} & & \Big\downarrow{\scriptstyle \text{div}} & & \Big\downarrow{\scriptstyle \text{div}} & & \Big\downarrow{\scriptstyle \text{div}} \\
P^0 & \xrightarrow{\text{grad}} & \delta_n^F & \xrightarrow{\text{curl}} & \delta_t^E & \xrightarrow{\text{div}} & \delta^V
\end{array}
$$

with distributional matrix-valued finite element spaces

$$
\begin{aligned}
V_{cc} &:= \{\gamma \in \mathcal{R}eg_0 + P^1 \delta_{nn}^F : \operatorname{curl}_E \gamma = 0\} \\
V_{cd} &:= \{\eta \in \mathcal{M}CS_0 + \mathcal{R}T_0 \otimes \delta_n^F : \operatorname{div}_E \eta = 0\} \\
V_{dd} &= \{\sigma \in P^0 \delta_{tt}^F + P^1 \delta_{tt}^E : \operatorname{div}_V \sigma = 0\}
\end{aligned}
$$

A different view on mixed methods for Poisson

Solve $(\nabla u, \nabla v)_{L_2} = f(v)$ with discontinuous $u_h \in P^k$.

Then

$$\nabla u_h = \sum_T \nabla_T u_h + \sum_F [u_h] \delta_n^F$$

and a direct evaluation of $(\nabla u_h, \nabla v_h)_{L_2}$ is not allowd.

Mimicing

$$\|\nabla u\|_{L_2} = \sup_{\sigma \in H(\mathrm{div})} \frac{(\nabla u, \sigma)_{L_2}}{\|\sigma\|_{L_2}}$$

on the discrete, i.e.

$$\|\nabla u_h\|_{L_2,h} = \sup_{\sigma_h \in \mathcal{R}T_k} \frac{\langle \nabla u_h, \sigma_h \rangle}{\|\sigma_h\|_{L_2}}$$

leads to a discrete L_2-like norm and inner product.

A different view on mixed methods for Poisson

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$$\|\nabla u\|_{L_2} = \sup_{\sigma \in H(\mathrm{div})} \frac{(\nabla u, \sigma)_{L_2}}{\|\sigma\|_{L_2}}$$

on the discrete, i.e.

$$\|\nabla u_h\|_{L_2,h} = \sup_{\sigma_h \in \mathcal{R}T_k} \frac{\langle \nabla u_h, \sigma_h \rangle}{\|\sigma_h\|_{L_2}}$$

leads to a discrete L_2-like norm and inner product.

This is exactly the mixed method in $H(\mathrm{div}) \times L_2$.

The HHJ-method for plates is the same trick to define $(\nabla^2 w_h, \nabla^2 v_h)_{L_2,h}$ for C^0-continuous finite elements.

$H(dd)$ and $H(cd)$-based methods for Elasticity and Stokes

Find stress $\sigma \in V_{dd}^k$ and displacement $u \in \mathcal{N}^k$ (the TDNNS method: robust for thin structers)

$$\begin{array}{lcrcll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, u \rangle & = & 0 & \forall\, \tau \in V_{dd} \\ \langle \operatorname{div} \sigma, v \rangle & & & = & f(v) & \forall\, v \in \mathcal{N} \end{array}$$

Astrid Pechstein (aka Sinwel) Phd-thesis and joint work ['11,'12,'18,'21]

Find $\sigma \in V_{cd}^k$, $u \in \mathcal{BDM}^k$, and $p \in P^{k-1}$ (the pressure-robust MCS method for Stokes):

$$\begin{array}{rcrcll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, u \rangle + (\operatorname{div} u, q) & = & 0 & \forall\, \tau \in V_{cd}, \forall\, q \in P^{k-1} \\ \langle \operatorname{div} \sigma, v \rangle + (\operatorname{div} v, p) & & & = & f(v) & \forall\, v \in \mathcal{BDM}^k \end{array}$$

Philip Lederer Phd-thesis and P. Lederer-J. Gopalakrishnan-JS ['20, '20]

$H(dd)$ methods for plates

Hellan-Herrmann-Johnson (HHJ) method for the Kirchhoff plate: ['60s and '70s, Arnold+Brezzi '85, I. Comodi '89]

Find bending moments $\sigma \in V_{dd}^k$ and vertical deflection $w \in \mathcal{L}^{k+1}$:

$$\begin{array}{lclcll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, \nabla w \rangle & = & 0 & \forall \tau \in V_{dd}^k \\ \langle \operatorname{div} \sigma, \nabla v \rangle & & & = & f(v) & \forall v \in \mathcal{L}^{k+1} \end{array}$$

Combination of HHJ and TDNNS for Reissner Mindlin [A. Pechstein-JS '17]:

Find $\sigma \in V_{dd}^k$ and $w \in \mathcal{L}^{k+1}$, $\beta \in \mathcal{N}^k$:

$$\begin{array}{lclcll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, \beta \rangle & = & 0 & \forall \tau \in V_{dd}^k \\ \langle \operatorname{div} \sigma, \delta \rangle & - & \frac{1}{t^2}(\nabla w - \beta, \nabla v - \delta) & = & f(v) & \forall v \in \mathcal{L}^{k+1}, \ \forall \delta \in \mathcal{N}^k, \end{array}$$

Free of locking, and for $t \to 0$ the discrete RM solution converges to the Kirchhoff solution.

The TDNNS mixed method for elasticity

The elasticity problem is equivalent to the mixed problem: Find $\sigma \in H(\operatorname{div}\operatorname{div})$ and $u \in H(\operatorname{curl})$ such that for tangentially continuous v and normal-normal continuous τ:

$$\begin{array}{ccccl} \int A\sigma : \tau & + & \sum_T \left\{ \int_T \operatorname{div}\tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} & = & 0 \qquad \forall\, \tau \\ \sum_T \left\{ \int_T \operatorname{div}\sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} & & & = & -\int f \cdot v \quad \forall\, v \end{array}$$

Proof: The second line is equilibrium, plus tangential continuity of the normal stress vector:

$$\sum_T \int_T (\operatorname{div}\sigma + f) v + \sum_E \int_E [\sigma_{n\tau}] v_\tau = 0 \qquad \forall\, v$$

Since the space requires continuity of σ_{nn}, the normal stress vector is continuous.
Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_E \int_E \tau_{nn} [u_n] = 0 \qquad \forall\, \tau$$

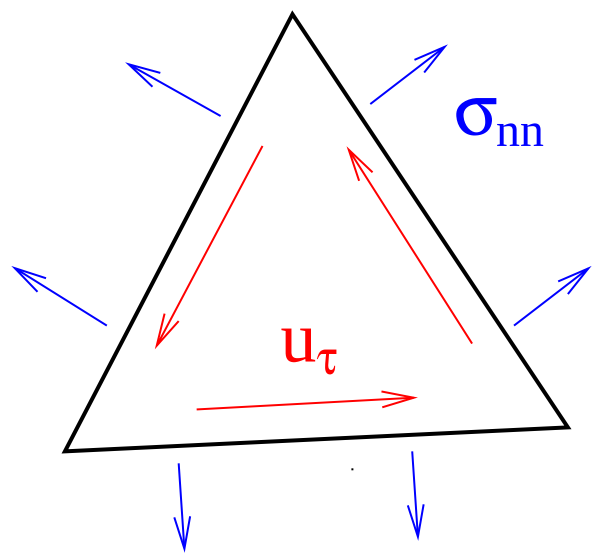
This is the constitutive relation, plus normal-continuity of the displacement. Tangential continuity of the displacement is implied by the space $H(\operatorname{curl})$.

Lowest order simplicial finite elements for TDNNS

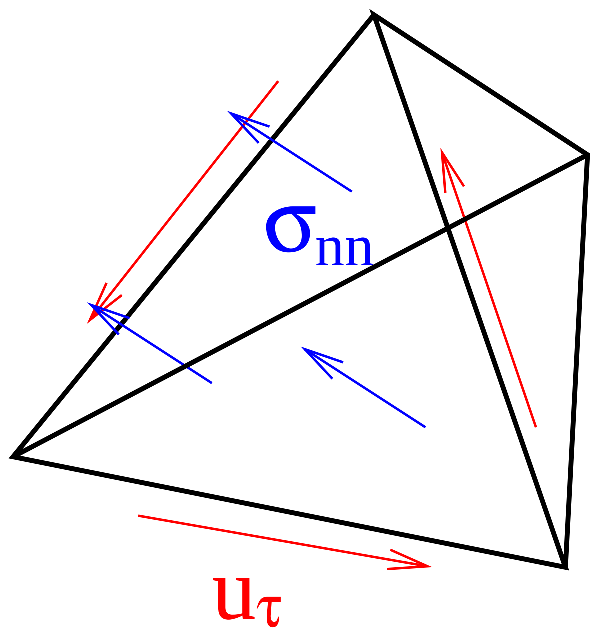
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

Triangular Finite Element:

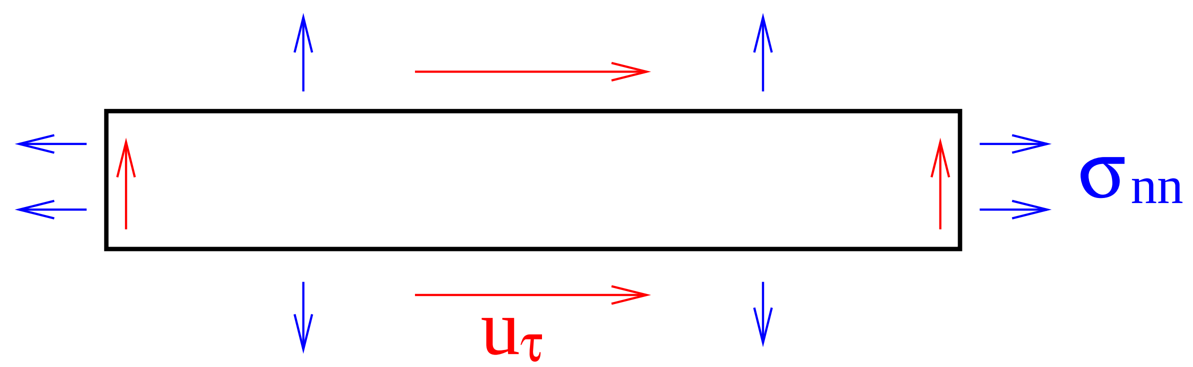


Tetrahedral Finite Element:



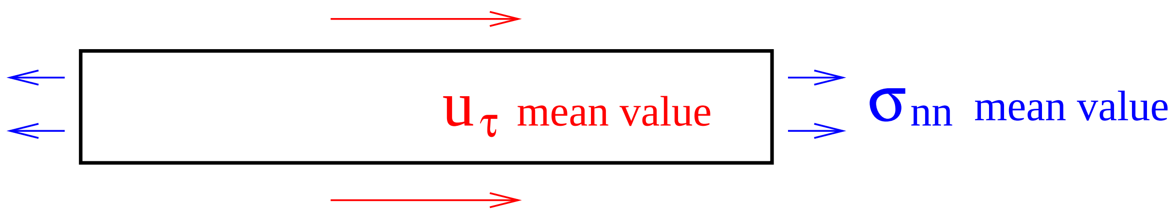
The quadrilateral element

Dofs for general quadrilateral element:

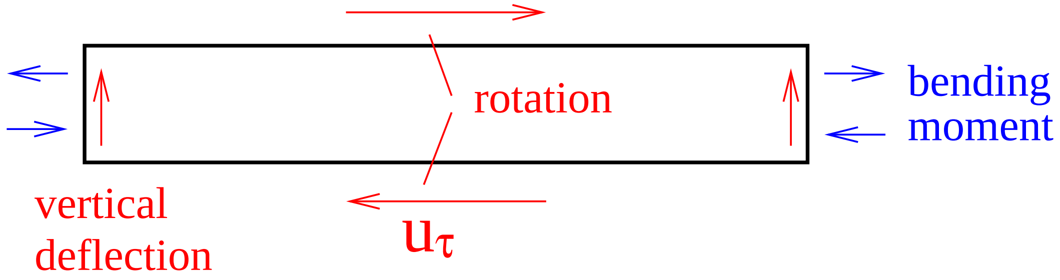


Thin beam dofs ($\sigma_{nn} = 0$ on bottom and top):

Beam stretching components:



Beam bending components:

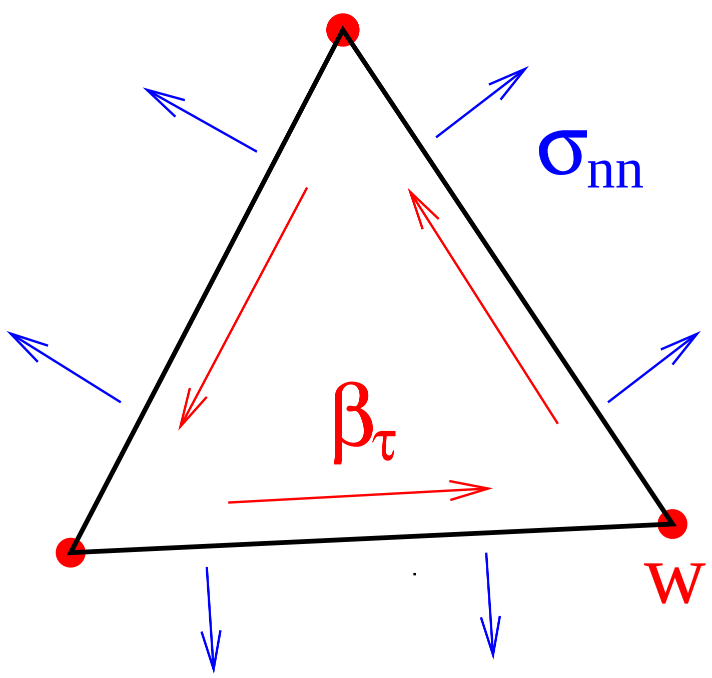


Reissner Mindlin Plates and Thin 3D Elements

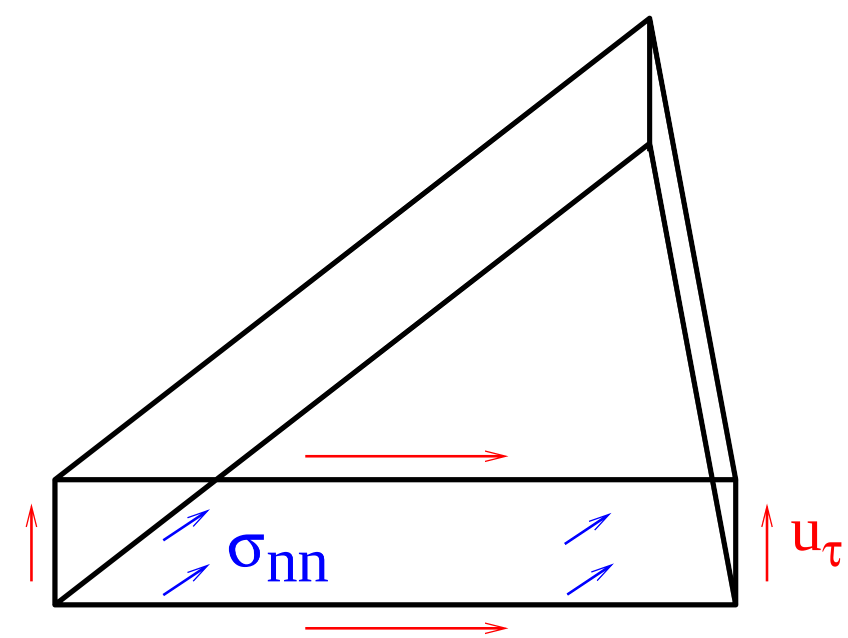
Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \operatorname{div}\sigma, \beta\rangle - t^{-2}\|\nabla w - \beta\|^2$$

Reissner Mindlin element:

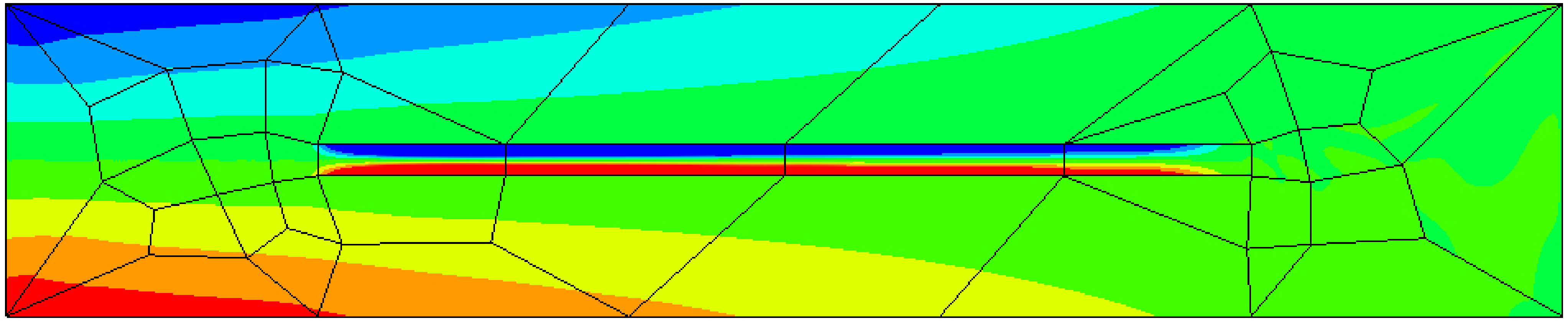


3D prism element:

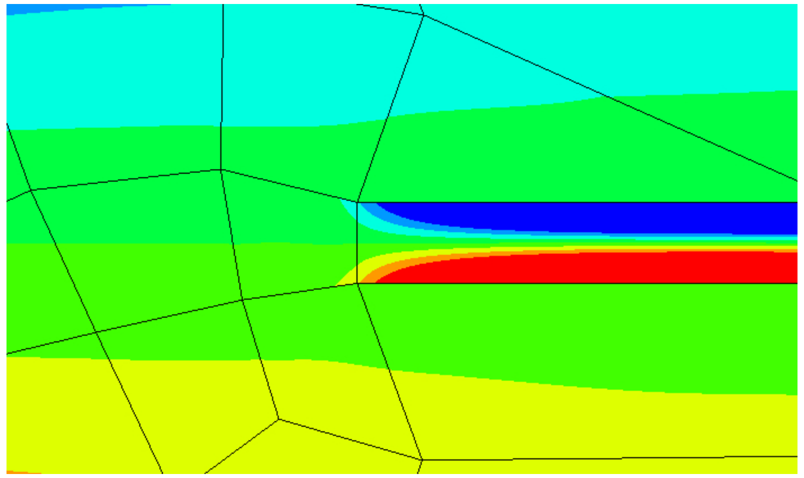


Hierarchical modeling: 3D discretization contains 2D reduced model

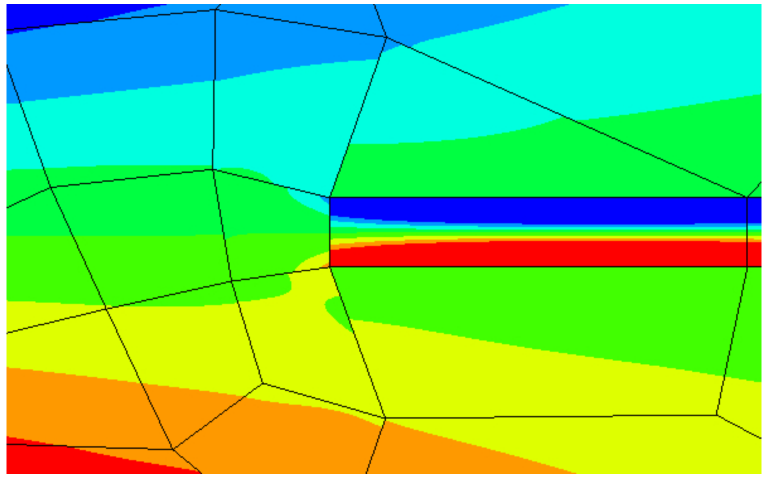
A beam in a beam



Reinforcement with $E = 50$ in medium with $E = 1$.



TDNNS mixed FEM, $p = 2$



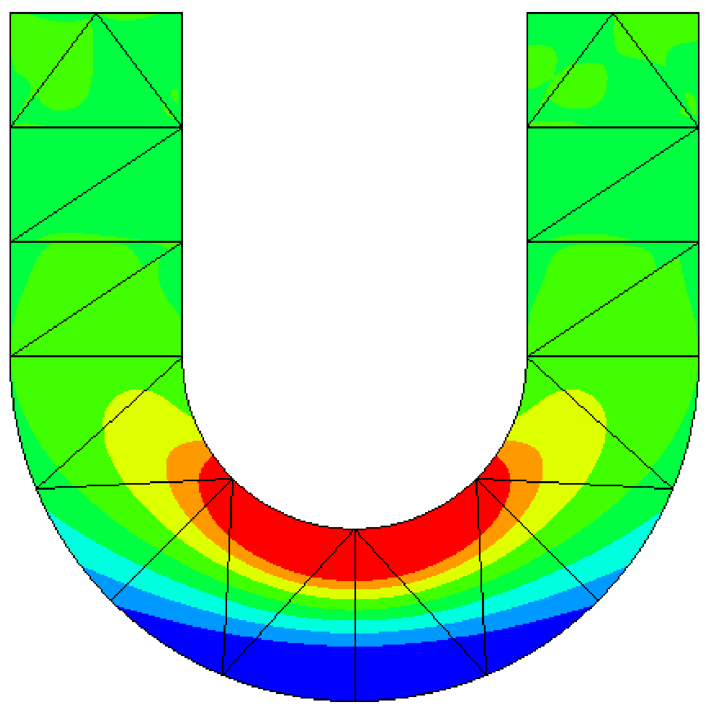
Primal FEM, $p = 3$

stress component σ_{xx}

Curved elements

fixed left top, pull right top

Elements of order 5



σ_{xx}

Mapped elements by two-sided Piola:

$$\sigma(x) = \frac{1}{J^2} F \widehat{\sigma}(\hat{x}) F^t$$

Mapping preserves nn-continuity, but not nt-continuity

$\operatorname{div} \sigma$ is not an algebraic transformation of $\widehat{\operatorname{div}} \hat{\sigma}$, but

$$\operatorname{div} \sigma = \frac{1}{J} F \, \widehat{\operatorname{div}} \, \hat{\sigma} + something(\nabla F) : \hat{\sigma}$$

Geometric nonlinear Elasticity

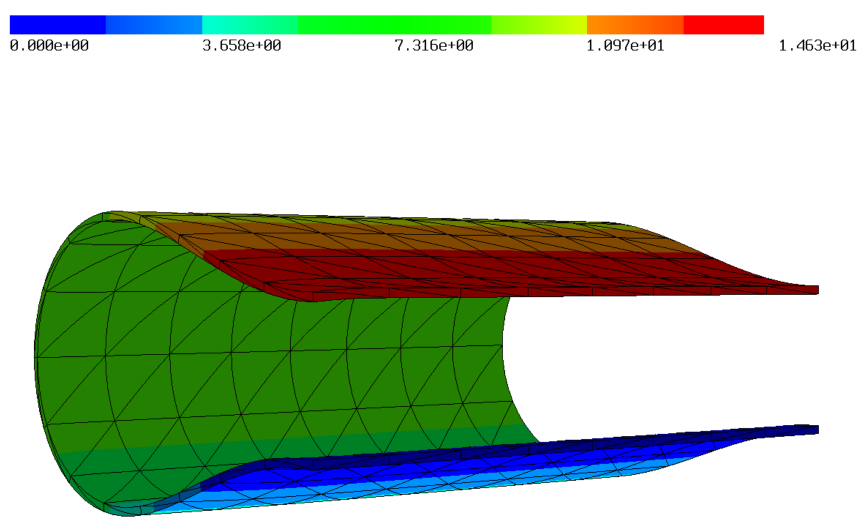
[M. Neunteufel + A. Pechstein + J.S '21, Phd-thesis M. Neunteufel 2021]

Hu-Washizu three-field mixed formulation

$$\min_{\substack{u,C \\ \langle C(u)-C,\Sigma\rangle=0}} \int_\Omega W(C)\,dx - \int_\Omega f u\,dx$$

with

- $u \in H(\mathrm{curl})$
- $\Sigma \in H(\operatorname{div}\operatorname{div})$ ... 2^{nd} Piola-Kirchhoff
- $C \in L_2(\mathbb{S})$ ... Cauchy-Green strain
- $W(.,)$ ... hyperelastic energy functional
- pressure-robust nearly incompressible $(\det F = 1)$



Riemann curvature and Incompatibility

The Kröner complex [Kröner 85, Int. J. Solid Structures]:

linear elasticity:

$$[H^1]^3 \xrightarrow{\varepsilon(\cdot)} H(cc) \xrightarrow{\text{inc}} H(dd) \xrightarrow{\text{div}} [H^{-1}]^3$$

nonlinear elasticity: Cauchy-Green strain, Riemann curvature and covariant divergence:

$$[H^1]^3 \xrightarrow{C(\cdot)} H(cc) \xrightarrow{R(\cdot)} H(dd) \xrightarrow{\text{div}_C} [H^{-1}]^3$$

with

$$\begin{aligned} C(\varphi) &= \nabla\varphi^T \nabla\varphi \\ R_{qijk}(g) &= \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} \end{aligned}$$

with Christoffel symbols Γ.

Riemann curvature for metric approximated by $H(\text{curl}\,\text{curl})$ fe: [M. Neunteufel, J. Gopalakrishnan, JS, M. Wardetzky, 23, 24]

Onging research

- Solving 3+1 Einstein equation for the metric in $H(\operatorname{curl}\operatorname{curl})$
  See talk by Edoardo Bonetti on Tue for the linearized Einstein-Bianchi equations
- Extension to the n-dimensional case. Heavy input from representation theory (Young tableaux)
- Equilibrated residual error estimation.
- AMG solvers

NGSolve examples using the presented spaces are found online at

`https://jschoeberl.github.io/talk-pdesoft/talk_pdesoft.html`